WIENER FILTERING

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INTRODUCTION

- Noise is present in many situations of daily life for ex: Microphones will record noise and speech.
- Goal: Reconstruct original signal
- Wiener filtering is a method to estimate the original signal as close as possible from the signals degraded by additive white noise
- Wiener Filter is the one which is based on the Linear minimum mean square error(LMMSE).
- Calculation of the Wiener filter requires the assumption that the signal and noise processes are second-order stationary. For this description, the data is WSS with zero mean will be considered .

Wiener filtering

There are three main problems that we will study

Smoothing

Filtering

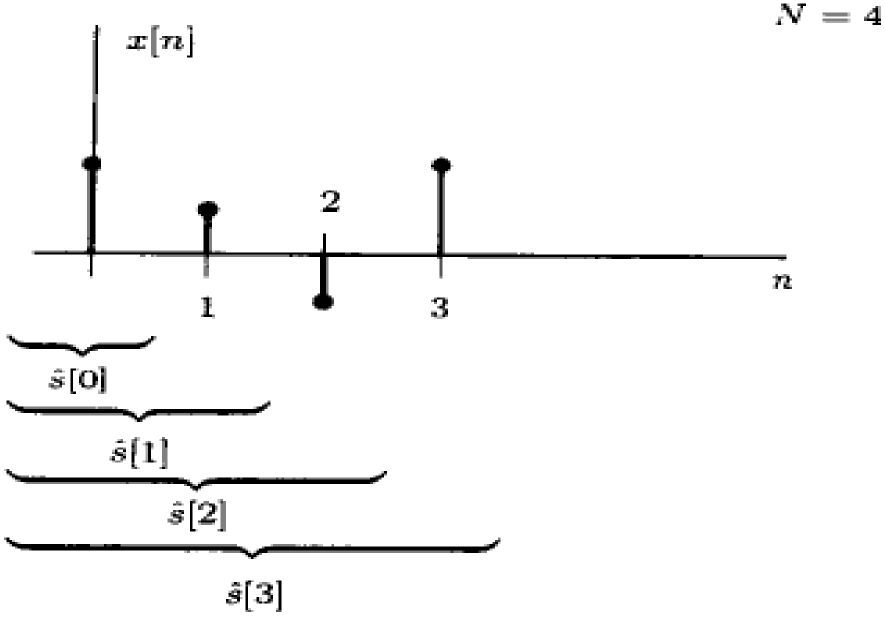
Prediction

to solve all three above problems we use

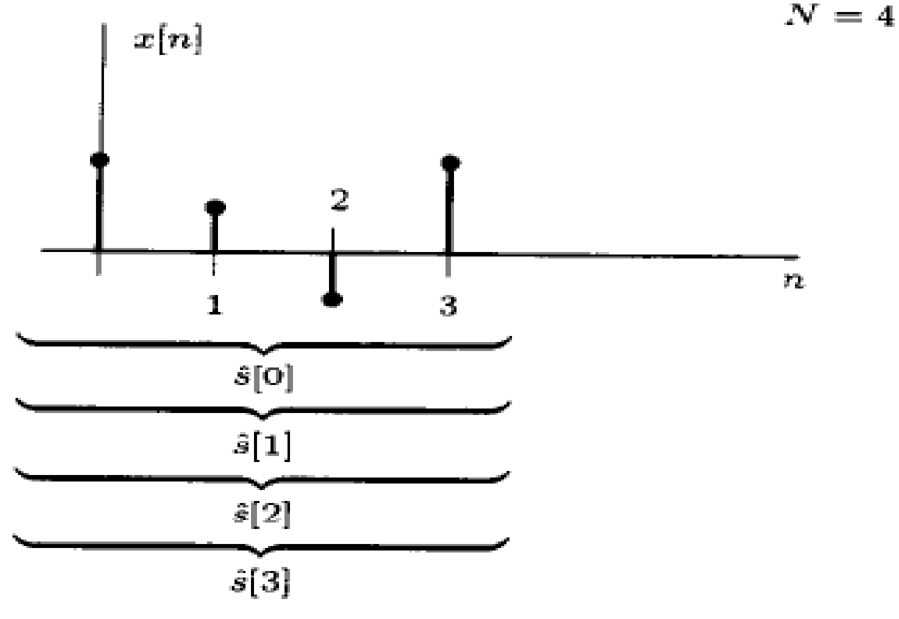
$$\hat{\Theta} = C_{\Theta x}\ C_{xx}^{-1}\ x$$

and the minimum MSE matrix given by

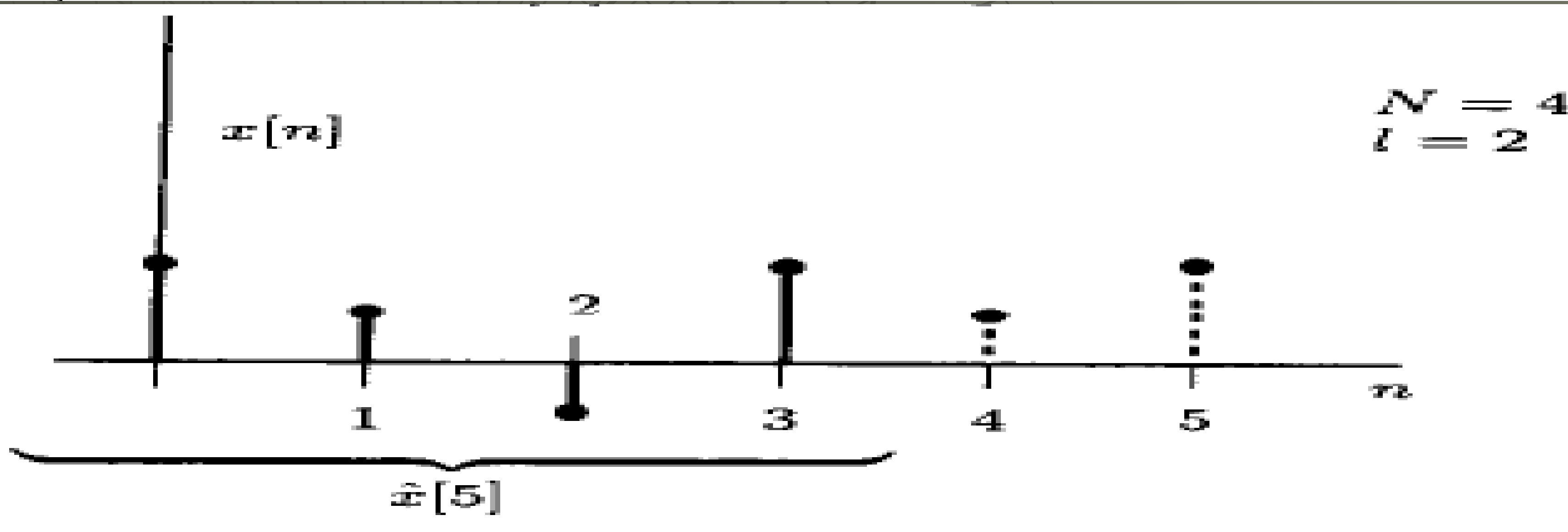
$$M_{\hat{\theta}} = C_{\theta\theta} - C_{\Theta x}\ C_{xx}^{-1}\ C_{x\theta}$$



(a) Filtering



(b) Smoothing



(c) Prediction

SMOOTHING

- Θ=s[n] is to be estimated for n=0,1,………N-1 based on the data set { x[0],x[1],…….x[N-1] } , where x[n]= s[n] + w[n].

- In this an estimation can not be obtained until all the data has been collected

$$\mathbf{C}_{xx} = \mathbf{R}_{xx} = \mathbf{R}_{ss} + \mathbf{R}_{ww}.$$

$$\mathbf{C}_{\theta x} = E(\mathbf{s}\mathbf{x}^T) = E(\mathbf{s}(\mathbf{s}+\mathbf{w})^T) = \mathbf{R}_{ss}.$$

Then the wiener estimation of the signal is

$$\hat{\mathbf{s}} = \mathbf{R}_{ss}(\mathbf{R}_{ss} + \mathbf{R}_{ww})^{-1}\mathbf{x}.$$

The corresponding minimum MSE matrix is

$$\begin{aligned}\mathbf{M}_{\hat{s}} &= \mathbf{R}_{ss} - \mathbf{R}_{ss}(\mathbf{R}_{ss} + \mathbf{R}_{ww})^{-1}\mathbf{R}_{ss} \\ &= (\mathbf{I} - \mathbf{W})\mathbf{R}_{ss}.\end{aligned}$$

Where $W = R_{ss}(R_{ss} + R_{ww})^{-1}$ is referred to as the wiener smoothing matrix

If N=1, we would estimate s[0] based on X[0]=S[0]+w[0]. Then, the wiener smoother W is given by

$$W = r_{ss}[0] \,/\, (r_{ss}[0] + r_{ww}[0])$$

$$= \eta / (\eta + 1)$$

where $\eta = r_{ss}[0] / r_{ww}[0]$ is the SNR

For high SNR so what $W \rightarrow 1$, we have $\hat{S}[0] \rightarrow X[0]$,

while for a low SNR so what $W \rightarrow 0$, we have $\hat{S}[n] \rightarrow 0$.

The corresponding minimum MSE is

$$M_{\hat{s}} = (1 - W)\, r_{ss}[0]$$

$$= \left(1 - \eta / (\eta + 1)\right) r_{ss}[0]$$

which for these two extremes is either 0 for a high SNR , $r_{xx}[0]$ for a low SNR

FILTERING

In this we estimate $\theta = s[n]$ based on $x[m]=s[m]+w[m]$ for $m=0,1,2\ldots\ldots n$.

The above problem is to filter the signal from noise, the signal sample is estimated based on the present and past data only.

$$C_{xx} = R_{ss} + R_{ww}$$

Also

$$\begin{aligned} \mathbf{C}_{\theta x} &= E\left(s[n]\left[\begin{array}{cccc} x[0] & x[1] & \ldots & x[n] \end{array}\right]\right) \\ &= E\left(s[n]\left[\begin{array}{cccc} s[0] & s[1] & \ldots & s[n] \end{array}\right]\right) \\ &= \left[r_{ss}[n]\, r_{ss}[n-1] \ldots r_{ss}[0]\right]. \end{aligned}$$

Then the estimator of the signal is

$$\hat{S}[n] = r'^{\,T}_{ss} \, (R_{ss} + R_{ww})^{-1} x$$

$\hat{s}[n] = a^T x$

where $a = (R_{ss} + R_{ww})^{-1} r'_{ss}$

To make the filtering correspondence we let $h^{(n)}[k] = a_{n-k}$

Then

$$\begin{aligned}\hat{s}[n] &= \sum_{k=0}^{n} a_k x[k] \\ &= \sum_{k=0}^{n} h^{(n)}[n-k]x[k]\end{aligned}$$

$$\hat{s}[n] = \sum_{k=0}^{n} h^{(n)}[k]x[n-k]$$

where $h^{(n)}[k]$ is the time varying FIR filter

To find the impulse response h we note that since

$$(\mathbf{R}_{ss} + \mathbf{R}_{ww})\,\mathbf{a} = \mathbf{r}'_{ss}$$

$$(\mathbf{R}_{ss} + \mathbf{R}_{ww})\,\mathbf{h} = \mathbf{r}_{ss}$$

Written out, the set of linear equations becomes

$$\begin{bmatrix} r_{xx}[0] & r_{xx}[1] & \cdots & r_{xx}[n] \\ r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[n-1] \\ \vdots & \vdots & \ddots & \vdots \\ r_{xx}[n] & r_{xx}[n-1] & \cdots & r_{xx}[0] \end{bmatrix} \begin{bmatrix} h^{(n)}[0] \\ h^{(n)}[1] \\ \vdots \\ h^{(n)}[n] \end{bmatrix} = \begin{bmatrix} r_{ss}[0] \\ r_{ss}[1] \\ \vdots \\ r_{ss}[n] \end{bmatrix}$$

These are the wiener-Hopf filtering equations

For large enough n it can be shown that the filter becomes time invariant, the Wiener-Hopf filtering equations can be written as

$$\sum_{k=0}^{\infty} h[k] r_{xx}[l-k] = r_{ss}[l] \qquad l = 0, 1, \ldots \ .$$

The same set of equations result if we attempt to estimate s[n] based on the present and infinite past . This is termed the infinite wiener filter.

let $$\hat{s}[n] = \sum_{k=0}^{\infty} h[k] x[n-k]$$

And use the orthogonality principle.

Then, E[(S[n] - Ŝ[n]) x[n-l])] =0 ; l=0,1.......

Hence,

$$E\left(\sum_{k=0}^{\infty} h[k]x[n-k]x[n-l]\right) = E(s[n]x[n-l])$$

and therefore, the equations to be solved for the infinite wiener filter impulse response are

$$\sum_{k=0}^{\infty} h[k]r_{xx}[l-k] = r_{ss}[l] \qquad l = 0, 1, \ldots \; .$$

The smoothing estimator takes the form

$$\hat{S}[n] = \sum_{k=-\infty}^{\infty} a_k \, x[k]$$

And by letting $h[k]=a_{n-k}$ we have the convolution sum

$$\hat{S}[n] = \sum_{k=-\infty}^{\infty} h[k] \; x[n-k]$$

The wiener equation becomes

$$\sum_{k=-\infty}^{\infty} h[k] \; r_{xx}[l-k] = r_{ss}[\,l\,] \quad ; \; -\infty < l < \infty$$

The difference from the filtering case is that now the equations must be satisfied for all, and there is no constraint that h[k] must be causal.

Hence we can use Fourier transform techniques to solve for the impulse response then

$$H(f) = P_{ss}(f) / P_{xx}(f)$$

$$= P_{ss}(f) \,/\, \left(P_{ss}(f) + P_{ww}(f)\right)$$

If we define SNR as $\eta(f) = P_{ss}(f) / P_{ww}(f)$

Then the optimal filter frequency response becomes

$$H(f) = \eta(f) / (\eta(f)+1)$$

Clearly, the filter response satisfies $0 < H(f) < 1$, and the wiener smoother response is $H(f) \approx 0$ when $\eta(f) \approx 0$ and $H(f) \approx 1$ when $n(f) \to \infty$

PREDICTION

The prediction problem in which we estimate θ =X[N-1+l] for l≥1 based on **X.**

we use $\mathbf{C}_{xx}=\mathbf{R}_{xx}$ and

$$\begin{aligned}\mathbf{C}_{\theta x} &= E\left[x[N-1+l]\left[\begin{array}{llll} x[0] & x[1] & \ldots & x[N-1]\end{array}\right]\right] \\ &= \left[\begin{array}{llll} r_{xx}[N-1+l] & r_{xx}[N-2+l] & \ldots & r_{xx}[l]\end{array}\right].\end{aligned}$$

let the latter vector be denoted by r'^{T}_{xx}. Then

$$\hat{x}[N-1+l] = \mathbf{r}'^{T}_{xx}\mathbf{R}^{-1}_{xx}\mathbf{x}.$$

$$\mathbf{a} = \mathbf{R}^{-1}_{xx}\mathbf{r}'_{xx}$$

$$\hat{x}[N-1+l] = \sum_{k=0}^{N-1} a_k x[k].$$

let $h[N-k]=a_k$ to allow filtering interpretation

$$\hat{x}[N-1+l] = \sum_{k=0}^{N-1} h[N-k]x[k] = \sum_{k=1}^{N} h[k]x[N-k]$$

$R_{xx}h=r_{xx}$

where $\mathbf{r}_{xx} = [r_{xx}[l]\, r_{xx}[l+1] \dots r_{xx}[N-1+l]]^T$. In explicit form they become

$$\begin{bmatrix} r_{xx}[0] & r_{xx}[1] & \dots & r_{xx}[N-1] \\ r_{xx}[1] & r_{xx}[0] & \dots & r_{xx}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ r_{xx}[N-1] & r_{xx}[N-2] & \dots & r_{xx}[0] \end{bmatrix} \begin{bmatrix} h[1] \\ h[2] \\ \vdots \\ h[N] \end{bmatrix} = \begin{bmatrix} r_{xx}[l] \\ r_{xx}[l+1] \\ \vdots \\ r_{xx}[N-1+l] \end{bmatrix}$$

The minimum MSE for l-step linear predictor is

$$M_{\hat{x}} = r_{xx}[0] - \mathbf{r}'^T_{xx}\mathbf{R}^{-1}_{xx}\mathbf{r}'_{xx}$$

$$\begin{aligned}
M_{\hat{x}} &= r_{xx}[0] - \mathbf{r}'^T_{xx}\mathbf{a} \\
&= r_{xx}[0] - \sum_{k=0}^{N-1} a_k r_{xx}[N-1+l-k] \\
&= r_{xx}[0] - \sum_{k=0}^{N-1} h[N-k] r_{xx}[N-1+l-k] \\
&= r_{xx}[0] - \sum_{k=1}^{N} h[k] r_{xx}[k+(l-1)].
\end{aligned}$$

Assume that x[n] is an AR(1) process with ACF

$$r_{xx}[k] = \frac{\sigma_u^2}{1-a^2[1]}\left(-a[1]\right)^{|k|}$$

$$\hat{x}[N] = \sum_{k=1}^{N} h[k] x[N-k].$$

Let l=1 and we solve for the h[k].

$$\sum_{k=1}^{N} h[k]\; r_{xx}[m-k] = r_{xx}[m] \quad ; \quad m=1,2,\ldots\ldots,N$$

$$\sum_{k=1}^{N} h[k]\; (-a[1])^{|m-k|} = (-a[1])^{|m|} \quad ; \quad m=1,2,\ldots\ldots,N$$

On solving above equation we get

$$h[k] = \begin{cases} -a[1] & k=1 \\ 0 & k=2,3,\ldots,N. \end{cases}$$

The one step linear predictor is $\hat{x}[N] = -a[1]x[N-1]$

And the minimum MSE is

$$\begin{aligned} M_{\hat{x}} &= r_{xx}[0] - \sum_{k=1}^{N} h[k] r_{xx}[k] \\ &= r_{xx}[0] + a[1] r_{xx}[1] \\ &= \frac{\sigma_u^2}{1-a^2[1]} + a[1]\frac{\sigma_u^2}{1-a^2[1]}(-a[1]) \\ &= \sigma_u^2. \end{aligned}$$

Similarly, the l step predictor by solving

$$\sum_{k=1}^{N} h[k]\ r_{xx}[m-k] = r_{xx}[m+l-1] \quad ; \quad m=1,2,\ldots\ldots,N$$

substituting the ACF for an AR(1) process, this becomes

$$\sum_{k=1}^{N} h[k]\ (-a[1])^{|m-k|} = (-a[1])^{|m+l-1|} \quad ; \quad m=1,2,\ldots\ldots,N$$

Then the impulse response is

$$h[k] = \begin{cases} (-a[1])^{l} & k=1 \\ 0 & k=2,3,\ldots,N \end{cases}$$

The l step predictor is

$$\hat{x}[(N-1)+l] = (-a[1])^{l}\, x[N-1]$$

and the minimum MSE for l step predictor is

$$
\begin{aligned}
M_{\hat{x}} &= r_{xx}[0] - h[1] r_{xx}[l] \\
&= \frac{\sigma_u^2}{1-a^2[1]} - (-a[1])^l \frac{\sigma_u^2}{1-a^2[1]} (-a[1])^l \\
&= \frac{\sigma_u^2}{1-a^2[1]} \left(1 - a^{2l}[1]\right).
\end{aligned}
$$

The predictor decays to zero with increase in l, since $|a[1]|<1$. This is also reflected in the minimum MSE, which is smallest for l=1 and increases for larger one.

THANK YOU