Matched Filters for Binary and M-ary spaces

Presented by:-
Akshata Singh   Y8104003
G.S. Kesarwani Y8104023

Guided by : R. Hegde , IIT K

Topics covered

- Introduction of Matched filters
  - Development of detector
  - Performance of Matched filters
  - Performance of Generalized Matched filters
- Detection of known Signal in presence of noise using Matched filter detector
  - Performance analysis for Binary Case
  - Performance analysis for M-ary Case

Introduction of Matched Filter

- The problem here pertains to detection of known signal in Gaussian noise.
- Since we assume noise to be Gaussian therefore the resultant test statistic is linear function of the data.
- The detector evolved from the above assumptions is termed as Matched filter.

Development of Detector

- The criterion used for detection is Neyman –Pearson(NP)– criterion.

- However Bayesian risk criterion could also be used , only threshold and detection performance will differ.

- The detection problem is to distinguish between the hypotheses

$$\begin{aligned} \mathcal{H}_0 &: x[n] = w[n] & n = 0, 1, \ldots, N-1 \\ \mathcal{H}_1 &: x[n] = s[n] + w[n] & n = 0, 1, \ldots, N-1 \end{aligned}$$

The NP detector decides H_1 if the likelihood ratio L(x) exceeds the threshold

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} > \gamma$$

where $\mathbf{x} = [x[0]\, x[1] \ldots x[N-1]]^T$. Since

$$\begin{aligned} p(\mathbf{x}; \mathcal{H}_1) &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - s[n])^2\right] \\ p(\mathbf{x}; \mathcal{H}_0) &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} x^2[n]\right] \end{aligned}$$

we have

$$L(\mathbf{x}) = \exp\left[-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1} (x[n] - s[n])^2 - \sum_{n=0}^{N-1} x^2[n]\right)\right] > \gamma.$$

We decide H_1 if :-

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n] > \gamma'.$$

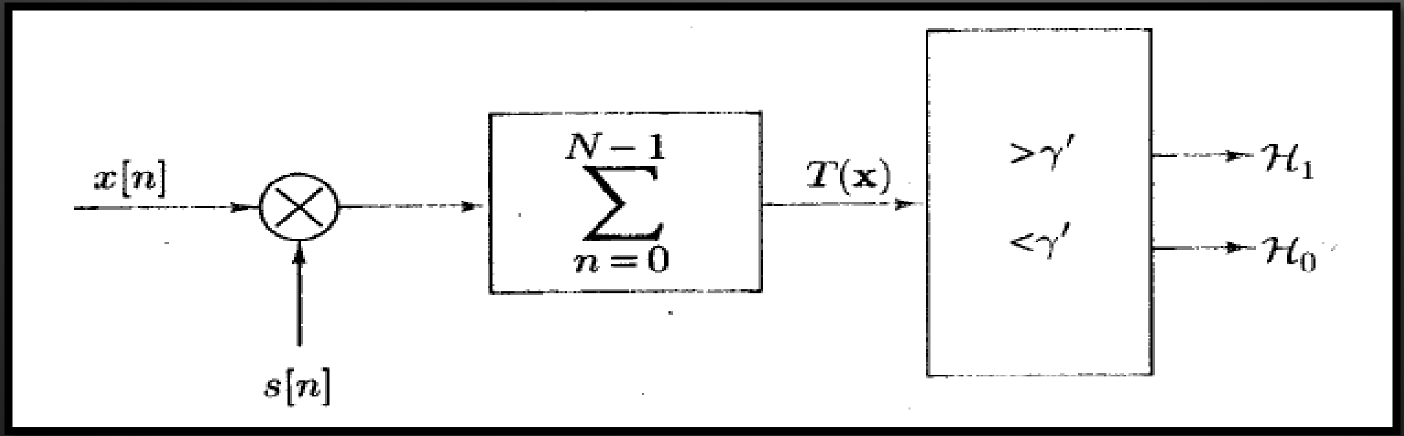
Where new Threshold is given by :-

$$\sigma^2 \ln \gamma + \frac{1}{2}\sum_{n=0}^{N-1} s^2[n].$$

This is the NP Detector which consists of test statistic T(x) and the threshold as given above.

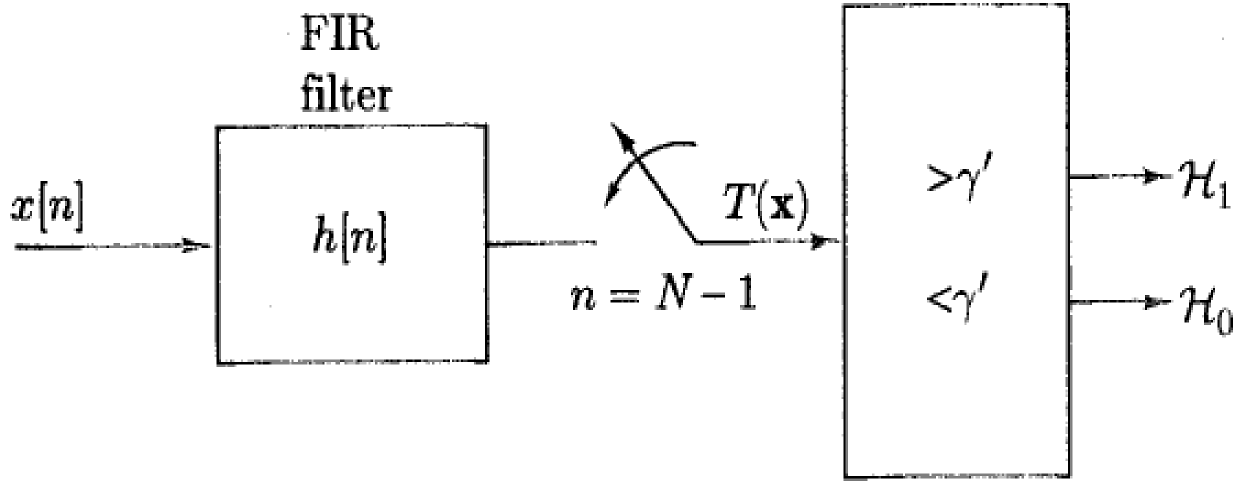
The above mentioned detector is referred to as a correlator since we correlate the received data with the replica of the signal.

The implementation of detector is shown below:



Alternatively we can implement the correlation process using FIR filter which is called Matched filter

The implementation of Matched filter is shown below:



$$h[n] = \begin{cases} s[N-1-n] & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

- The output at time n is

$$y[n] = \sum_{k=0}^{n} h[n-k]x[k]$$

- As we know

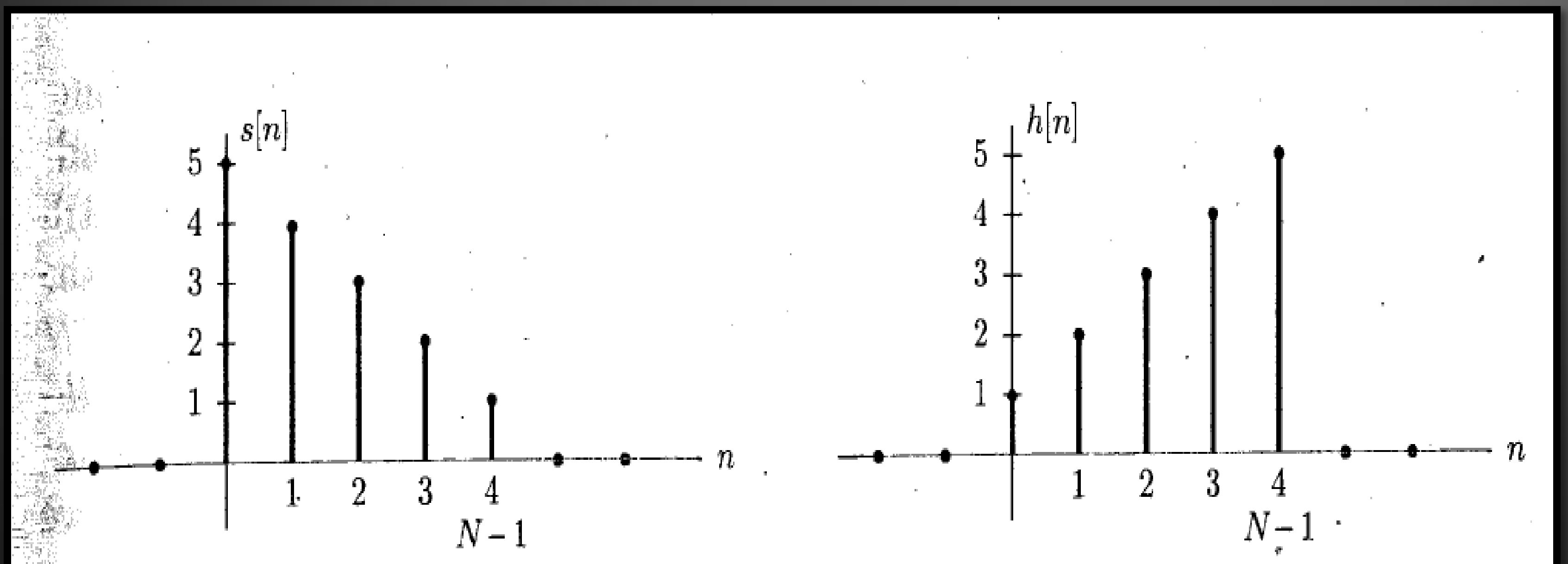
$$h[n] = s[N-1-n] \qquad n = 0, 1, \ldots, N-1$$

$$y[n] = \sum_{k=0}^{n} s[N-1-(n-k)]x[k].$$

- The output of the filter at time n= N-1 is

$$y[N-1] = \sum_{k=0}^{N-1} s[k]x[k]$$

Example of Matched filter impulse response



The Matched filter impulse response h(n) is obtained by flipping the signal s(n) about n=0 and shifting to the right by N-1 samples

- Since ,the signal output attains a maximum at the sampling time $n = N-1$,hence best detection performance will be obtained. (Assuming that the signal begins at $n=0$)

- However for signals with unknown arrival time we can not use the matched filter in its original form because of poor detection performance .

Properties of Matched Filter

- The matched filter output is just the signal energy when the noise is absent
- It maximizes the SNR at the output of the FIR filter which is given by :

$$
\begin{aligned}
\eta &= \frac{E^2(y[N-1];\mathcal{H}_1)}{\operatorname{var}(y[N-1];\mathcal{H}_1)} \\
&= \frac{\left(\sum_{k=0}^{N-1} h[N-1-k]s[k]\right)^2}{E\left[\left(\sum_{k=0}^{N-1} h[N-1-k]w[k]\right)^2\right]}
\end{aligned}
$$

Where y(N-1) is the output of the filter sampled at n= N-1

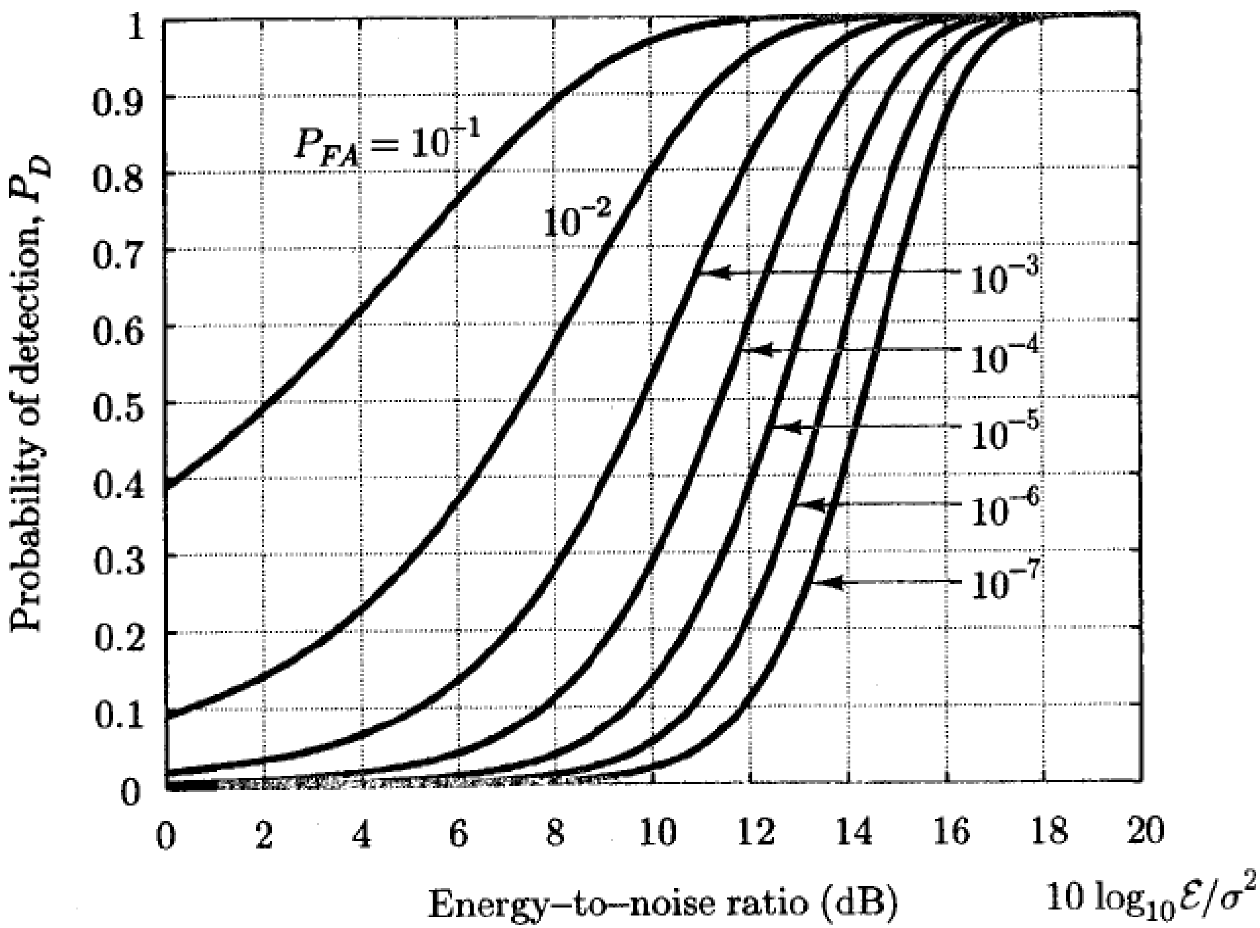
Performance of Matched filter

- The relation between probability of detection P_D for a given probability of false alarm P_{FA} is given as

$$
\begin{aligned}
P_D &= Q\left(\frac{\sqrt{\sigma^2\mathcal{E}}\,Q^{-1}(P_{FA})}{\sqrt{\sigma^2\mathcal{E}}} - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right) \\
&= Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right).
\end{aligned}
$$

- The Key parameter is the SNR (ϵ/σ^2)at the matched filter output ,as SNR increases P_D increases. The shape of the signal does not affect the detection performance.

Detection performance of Matched Filter



Generalized Matched Filter

- We have seen that Matched filter is optimum only when the Noise is WGN.

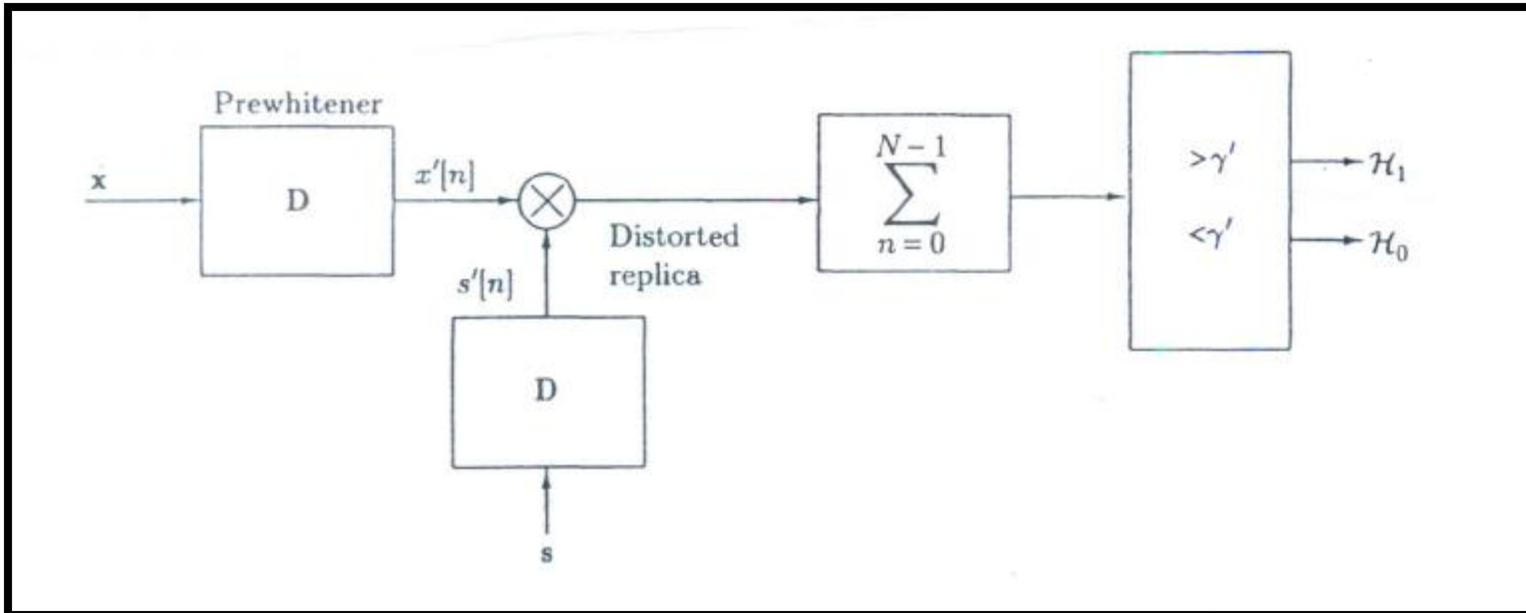
- When Noise is correlated (colored noise) but WSS then we can pre-whiten the received data and signal to achieve the better performance .

Generalised Matched filter

- We decide H1 if

$$\boxed{T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s} > \gamma'.}$$

- Where C is the Correlation matrix .
- Since C is positive definite then it can be shown that C^{-1} exists and is also positive definite therefore can be factored as $C^{-1} = D^T D$
- The Test Statistic becomes $T(x) = X^T D^T D s$
- Where D is pre-whitening Matrix.



- Generalised Matched filter as prewhitener and replica correlator

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}\right).$$

- In case of Generalized Matched filter the P_D increases monotonically with $s^T C^{-1} s$ not with SNR (ϵ/σ^2).

- Hence signal can be designed to maximize $s^T C^{-1} s$ and hence P_D

- .Signal shape is important in this case

Binary Case

- In communication systems we transmit one of M signals .The signals are known to the receiver but receiver has to decide which one.

- For Binary case we consider M=2

- For this case we have the following hypothesis testing problem.

  $H_0 : x(n) = s_0(n)+w(n) \quad n= 0,1,2\ldots\ldots,N-1$

  $H_1 : x(n) = s_1(n)+w(n) \quad n= 0,1,2\ldots\ldots,N-1$

Binary Case

- Here each type of error (Type I and Type II) is equally undesirable , the minimum probability of error criterion is chosen. We decide H1 if

$$\frac{p(\mathbf{x}|\mathcal{H}_1)}{p(\mathbf{x}|\mathcal{H}_0)} > \gamma = \frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)} = 1$$

- Here we have assumed equal prior probabilities of transmitting S_0 and S_1
- We choose the hypothesis having larger conditional likelihood.

$$p(\mathbf{x}|\mathcal{H}_i) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-s_i[n])^2\right]$$

- We decide H_i for which

$$D_i^2 = \sum_{n=0}^{N-1} (x[n] - s_i[n])^2$$

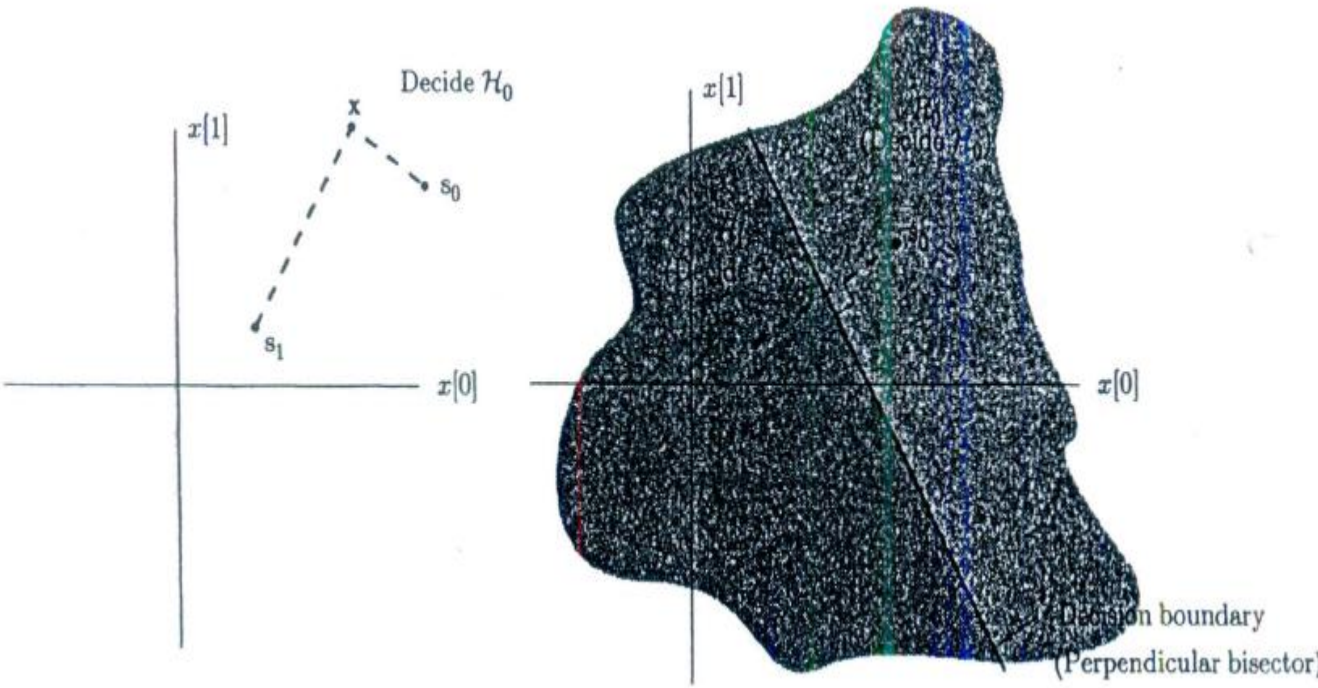
is Minimum

- This is referred to as a minimum distance receiver
- If we consider the data and signal samples in R^N then

$$\begin{aligned} D_i^2 &= (\mathbf{x} - \mathbf{s}_i)^T (\mathbf{x} - \mathbf{s}_i) \\ &= ||\mathbf{x} - \mathbf{s}_i||^2 \end{aligned}$$

- We choose the hypothesis whose signal vector is closest to X

- If we consider N=2 The decision region is decided by dividing the plane in to two regions which are separated by the perpendicular bisector of the line segment joining the two signal vector as shown below:-



The minimum distance receiver

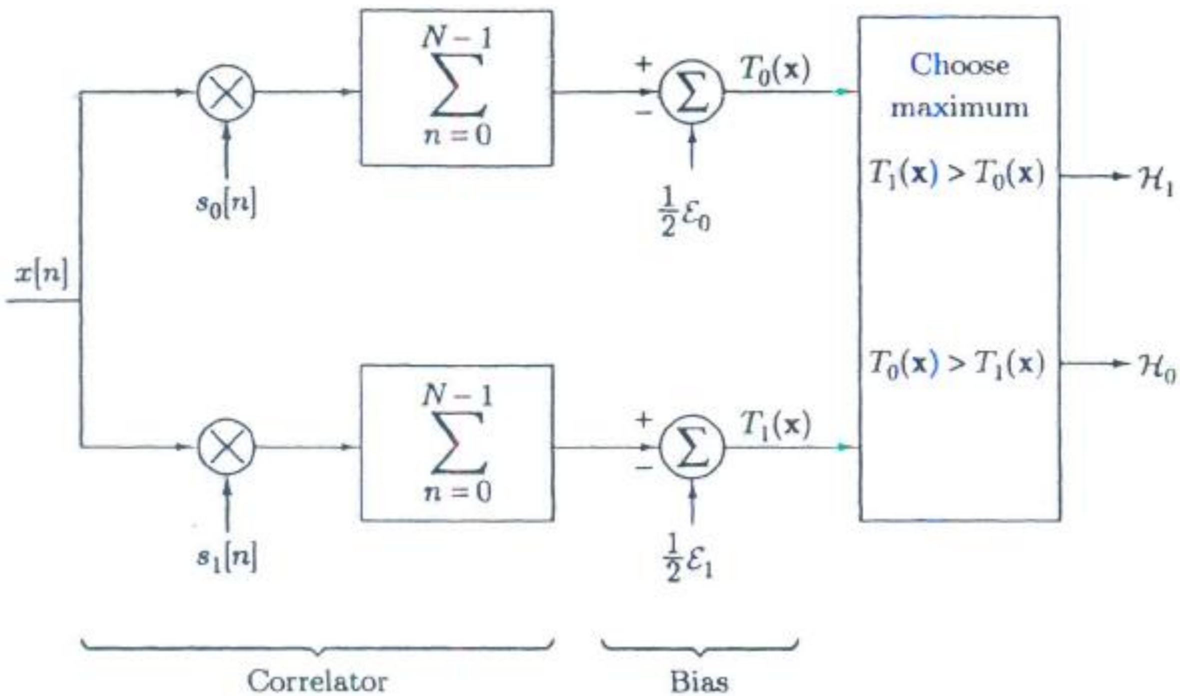
The minimum distance receiver can also be expressed in a more familiar form

$$D_i^2 = \sum_{n=0}^{N-1} x^2[n] - 2\sum_{n=0}^{N-1} x[n]s_i[n] + \sum_{n=0}^{N-1} s_i^2[n]$$

we decide $\mathcal{H}_i$ for which

$$\begin{aligned} T_i(\mathbf{x}) &= \sum_{n=0}^{N-1} x[n]s_i[n] - \frac{1}{2}\sum_{n=0}^{N-1} s_i^2[n] \\ &= \sum_{n=0}^{N-1} x[n]s_i[n] - \frac{1}{2}\mathcal{E}_i \end{aligned}$$

The minimum distance receiver



Performance Analysis for Binary Case

- We determine the P_e for the ML receiver for equal signal probabilities which is given by

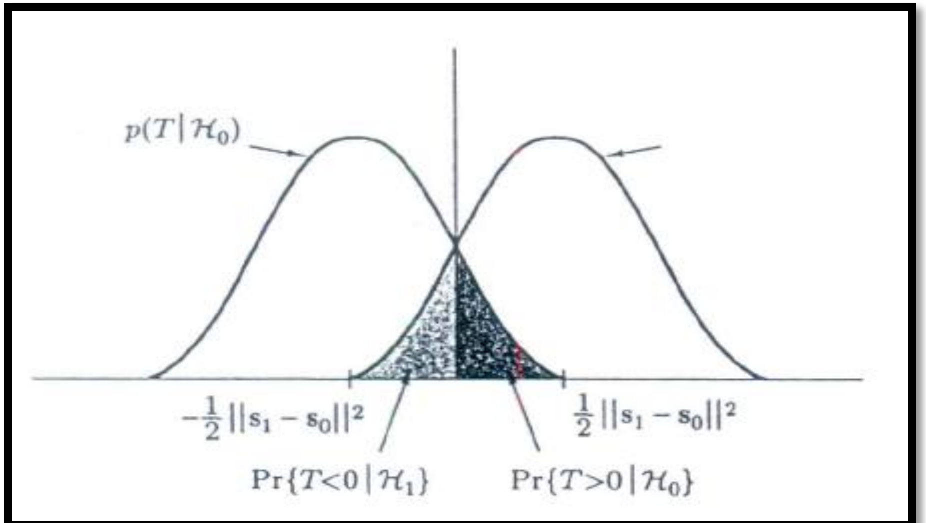
$$
\begin{aligned}
P_e &= \frac{1}{2}\left[P(\mathcal{H}_1|\mathcal{H}_0) + P(\mathcal{H}_0|\mathcal{H}_1)\right] \\
&= \frac{1}{2}\left[\Pr\{T_1(\mathbf{x}) > T_0(\mathbf{x})|\mathcal{H}_0\} + \Pr\{T_0(\mathbf{x}) > T_1(\mathbf{x})|\mathcal{H}_1\}\right] \\
&= \frac{1}{2}\left[\Pr\{T_1(\mathbf{x}) - T_0(\mathbf{x}) > 0|\mathcal{H}_0\} + \Pr\{T_0(\mathbf{x}) - T_1(\mathbf{x}) > 0|\mathcal{H}_1\}\right].
\end{aligned}
$$

Let $T(\mathbf{x}) = T_1(\mathbf{x}) - T_0(\mathbf{x})$. Then,

$$\boxed{T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n](s_1[n] - s_0[n]) - \frac{1}{2}(\mathcal{E}_1 - \mathcal{E}_0)}$$

- The Statistic is Gaussian random variable with

$$T \sim \mathcal{N}\left(-\frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_0\|^2, \sigma^2\|\mathbf{s}_1 - \mathbf{s}_0\|^2\right)$$



- The errors are same because of inherent receiver symmetry

$$P_e = \Pr\{T(\mathbf{x}) > 0|\mathcal{H}_0\}$$

$$P_e = Q\left(\frac{\frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_0\|^2}{\sqrt{\sigma^2\|\mathbf{s}_1 - \mathbf{s}_0\|^2}}\right)$$

or finally

$$P_e = Q\left(\frac{1}{2}\sqrt{\frac{\|\mathbf{s}_1 - \mathbf{s}_0\|^2}{\sigma^2}}\right).$$

- As $\|s_1 - s_0\|$ increases , P_e decreases as expected.
- However there is limitation in average power due to FCC regulations or system constraints.
- Average energy assuming equal prior probabilities

$$\bar{\mathcal{E}} = \frac{1}{2}(\mathcal{E}_0 + \mathcal{E}_1)$$

$$\begin{aligned} ||\mathbf{s}_1 - \mathbf{s}_0||^2 &= \mathbf{s}_1^T\mathbf{s}_1 - 2\mathbf{s}_1^T\mathbf{s}_0 + \mathbf{s}_0^T\mathbf{s}_0 \\ &= 2\bar{\mathcal{E}} - 2\mathbf{s}_1^T\mathbf{s}_0 \\ &= 2\bar{\mathcal{E}}(1-\rho_s) \end{aligned}$$

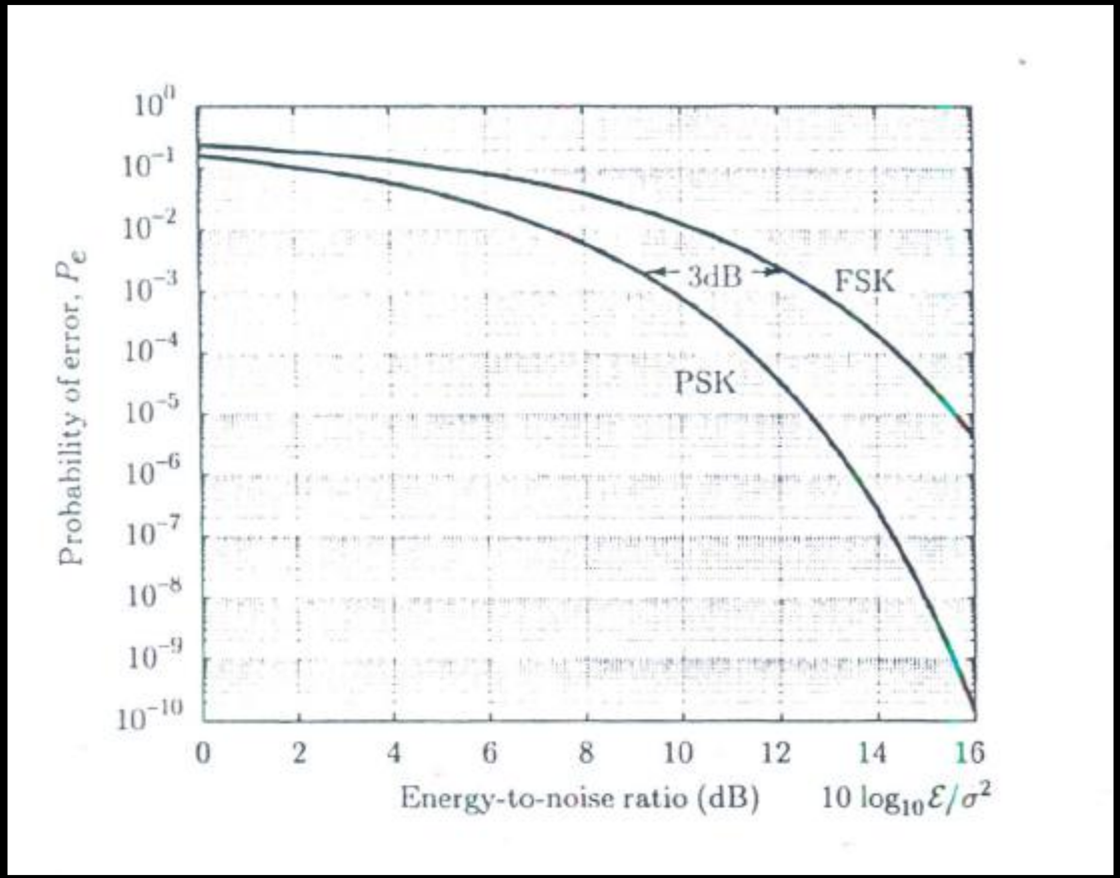
where we define ρ_s as

$$\rho_s = \frac{\mathbf{s}_1^T\mathbf{s}_0}{\frac{1}{2}(\mathbf{s}_1^T\mathbf{s}_1 + \mathbf{s}_0^T\mathbf{s}_0)}.$$

- Where ρ_s is signal correlation coefficient
- If the signal are chosen to be orthogonal

$$\boxed{P_e = Q\left(\sqrt{\frac{\bar{\mathcal{E}}(1-\rho_s)}{2\sigma^2}}\right).}$$

Performance curves



M-ary Case

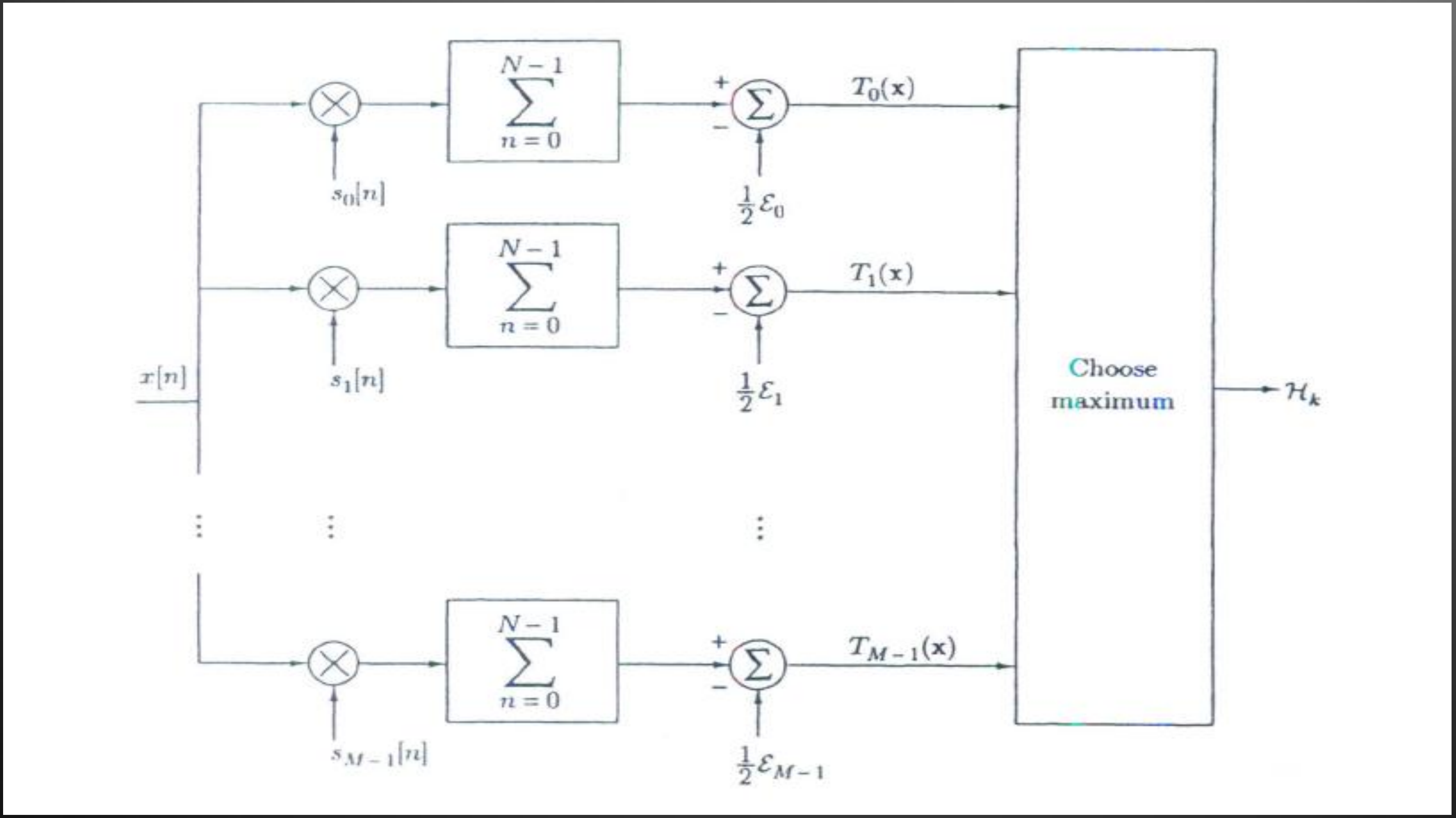
- Now if we transmit one of M signals with equal prior probabilities .
- The optimal receiver is again a minimum distance receiver and we choose H_k

$$\boxed{T_k(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s_k[n] - \frac{1}{2}\mathcal{E}_k}$$

- $T_k(x)$ is the maximum test statistic of

$$\{T_0(\mathbf{x}), T_1(\mathbf{x}), \ldots, T_{M-1}(\mathbf{x})\}$$

The optimal receiver



- An error occurs is any of the M−1 test statistic exceeds the one associated with true hypothesis.

- We consider signal to be orthogonal , then the statistics will be independent. Since joint Gaussian random variables with orthogonality assumption are uncorrelated hence independent.

- If the signal energies are same, an error is committed, if H_i is the true hypothesis but T_i is not maximum. Therefore

$$\boxed{P_e = \sum_{i=0}^{M-1} \Pr\{T_i < \max(T_0, \ldots, T_{i-1}, T_{i+1}, \ldots, T_{M-1}) | \mathcal{H}_i\} P(\mathcal{H}_i).}$$

- By symmetry all of the conditional probabilities in the above sum are same and hence

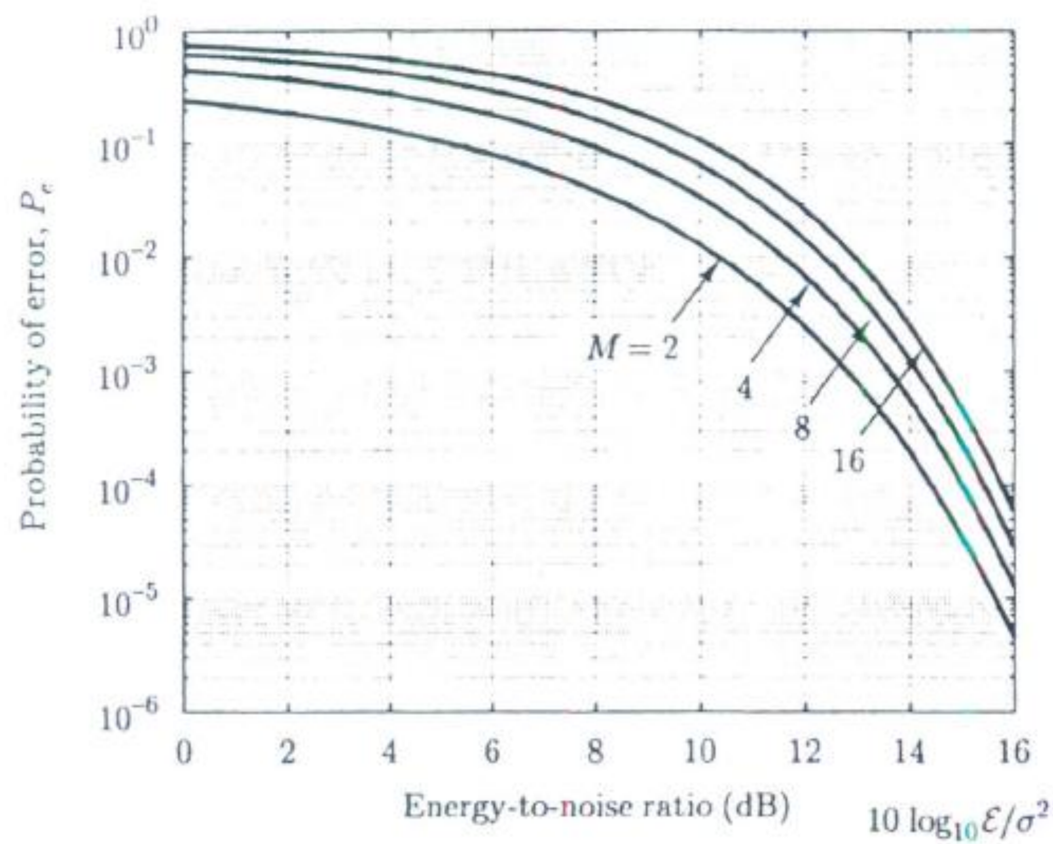
$$\boxed{P_e = \Pr\{T_0 < \max(T_1, T_2, \ldots, T_{M-1}) | \mathcal{H}_0\}.}$$

Therefore

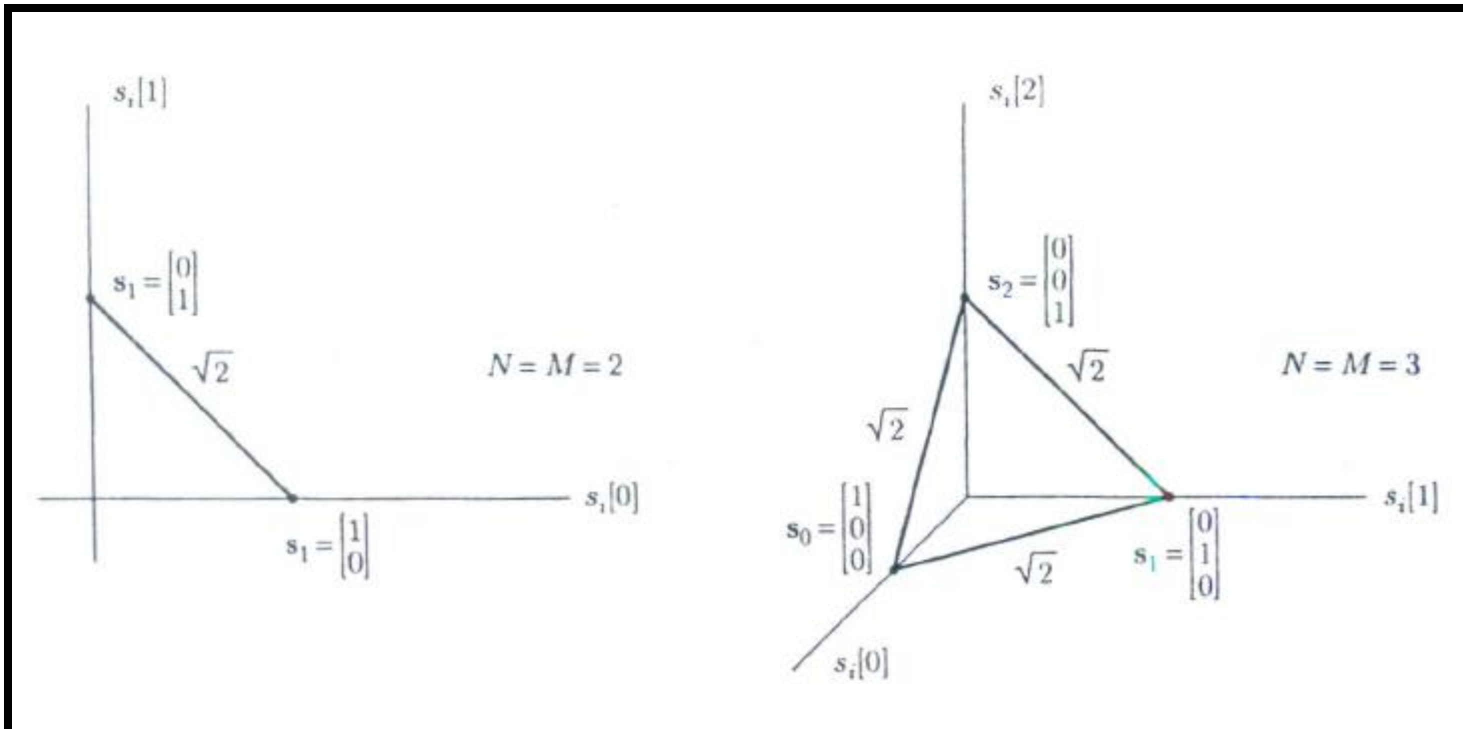
$$\begin{aligned} P_e &= 1 - \Pr\{T_0 > \max(T_1, T_2, \ldots, T_{M-1}) | \mathcal{H}_0\} \\ &= 1 - \Pr\{T_1 < T_0, T_2 < T_0, \ldots, T_{M-1} < T_0 | \mathcal{H}_0\} \\ &= 1 - \int_{-\infty}^{\infty} \Pr\{T_1 < t, T_2 < t, \ldots, T_{M-1} < t | T_0 = t, \mathcal{H}_0\} p_{T_0}(t) dt \\ &= 1 - \int_{-\infty}^{\infty} \prod_{i=1}^{M-1} \Pr\{T_i < t | \mathcal{H}_0\} p_{T_0}(t) dt \end{aligned}$$

$$P_e = 1 - \int_{-\infty}^{\infty} \Phi^{M-1}(u) \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(u - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right)^2\right] du$$

- The expression shows that P_e depends on the ENR (ϵ/σ^2).This dependence is plotted for various values of M :-



- Consider the cases of M=N=2 and M=N=3 orthogonal signals.
- The illustration of increasing Pe with increase in number of M is shown below:-



- Here the distance between the signals are the same for $M=2$ and $M=3$,each signal has energy $\epsilon=1$.

- As M increases we must choose from among a larger set of signals and therefore , P_e must increase with M

Thank You