Introduction to detection

eg. a) Radar detection   b) BPSK detection   c) Speech Detection

<u>Discussion</u> [PP. 1-7 KAY Vol.2]

Detection problem
- a) Binary Hypothesis testing
- b) Multiple Hypothesis testing

eg a) sinusoid with phase $\underline{0^\circ \text{ or } 180^\circ}$
b) Speech digit recognition (0-9)

Mathematical Detection problem

$$\begin{cases} \text{Hypothesis } 1: & x[0] = w[0] \quad ; \text{ Noise only} \\ \text{Hypothesis } 2: & x[0] = \underbrace{1}_{(A=1)} + w[0] \quad ; \text{ Signal + Noise} \end{cases}$$

<u>Detection</u>

$x[0] > 1/2$; signal present

$x[0] < 1/2$; Noise only

Note $E[x[0]] = 1$ for signal present ($\because A=1$)

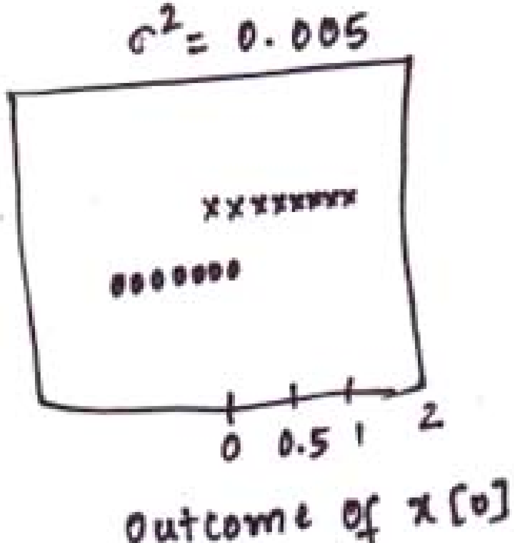
$E[x[0]] = 0$ for noise only

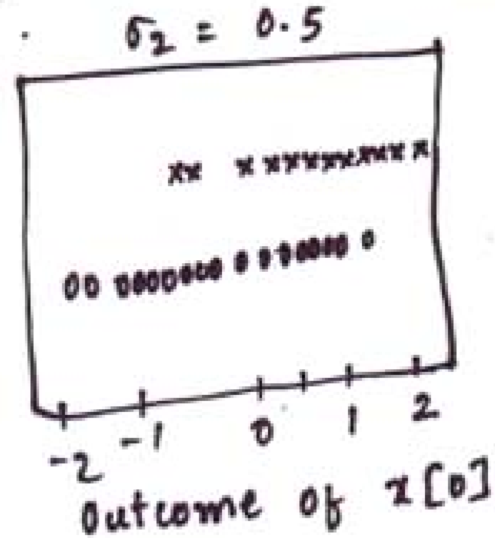
But an error can happen if

$w[0] < -1/2$; signal in noise present

$w[0] > 1/2$; noise only present

The error obviously depends on $\underline{\sigma}^2$ of $w[0]$





x = Signal + Noise

o = Noise

Histogram version in Fig 1.5

Pdf of noise

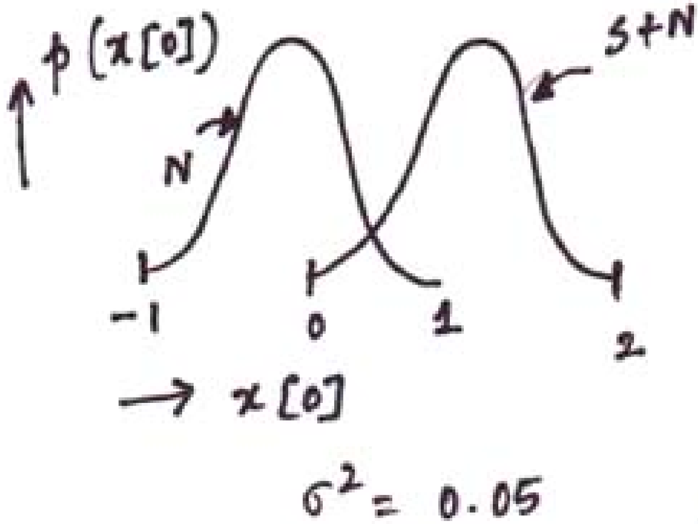
$$p(w[0]) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} w^2[0]\right)$$

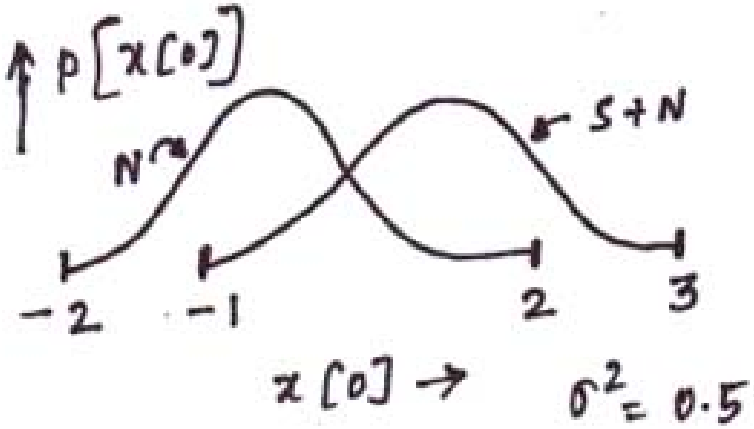
a) If $\sigma^2 = 0.05, 0.5$: NOISE ONLY CASE

$$p(x[0]) = \begin{cases} 1/\sqrt{0.1\pi} \exp\left(-10x^2[0]\right) ; & \sigma^2 = 0.05 \\ 1/\sqrt{\pi} \exp\left(-x^2[0]\right) ; & \sigma^2 = 0.5 \end{cases}$$

b) SIGNAL + NOISE CASE

$$p(x[0]) = \begin{cases} 1/\sqrt{0.1\pi} \exp\left(-10(x[0]-1)^2\right) ; & \sigma^2 = 0.05 \\ 1/\sqrt{\pi} \exp\left(-(x[0]-1)^2\right) ; & \sigma^2 = 0.5 \end{cases}$$





Pdf's of x[0] for N, S+N

* Detection performance improves as A^2/σ^2 (SNR) increases

- Formal Modeling of the Detection problem

$$\mathcal{H}_0 : x[0] = w[0]$$
$$\mathcal{H}_1 : x[0] = 1 + w[0]$$

Pdf^s under each hypothesis

$$p(x[0]; \mathcal{H}_0) = 1/\sqrt{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2} x^2[0]\right)$$
$$p(x[0]; \mathcal{H}_1) = 1/\sqrt{2\pi\sigma^2} \exp\left(-1/2\sigma^2 (x[0]-1)^2\right)$$

** Ask whether $x[0]$ is generated by $p(x[0]; \mathcal{H}_0)$ <u>or</u> $p(x[0]; \mathcal{H}_1)$.

-

Generalize to: A (Pdf parameterized by A)

$$p(x[0];A) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x[0]-A)^2\right)$$

Symbolic tests (Parameter test)

$$\boxed{\begin{aligned} \mathcal{H}_0 &: A = 0 \\ \mathcal{H}_1 &: A = 1 \end{aligned}}$$

$\because$ $p(x[0];\mathcal{H}_0)$ if $A=0$;
$p(x[0];\mathcal{H}_1)$ if $A=1$;

Assigning priors to $\mathcal{H}_0$ and $\mathcal{H}_1$ (say 1/2)

$$p(x[0]|\mathcal{H}_0) = 1/\sqrt{2\pi\sigma^2}\ \exp\left(-1/2\sigma^2\, x^2[0]\right)$$

$$p(x[0]|\mathcal{H}_1) = 1/\sqrt{2\pi\sigma^2}\ \exp\left(-1/2\sigma^2\,(x[0]-1)^2\right)$$

Asymptotics (Large data length records)

Estimation: Assume Asymptotic / High SNR

Detection: Success depends on large data length.

Let

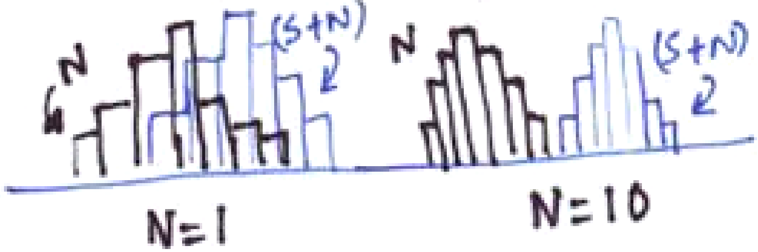
$$\mathcal{H}_0 : x[n] = w[n] \quad ; n = 0, 1, \ldots, (N-1)$$

$$\mathcal{H}_1 : x[n] = w[n] + A \; ; n = 0, 1, \ldots, (N-1)$$

where $w[n] \sim WGN(0, \sigma^2)$

Decide on $\mathcal{H}_1$ if $T = \frac{1}{N} \sum_{n=0}^{N-1} x[n] > \gamma$

As 'N' increases detection perf. increases



Quantify as deflection co-efficient d^2

$$d^2 = \frac{(E(T;\mathcal{H}_1) - E(T;\mathcal{H}_0))^2}{var(T;\mathcal{H}_0)}$$

'd' essentially quantifies overlap and

Note that $var(T;\mathcal{H}_0) = var(T;\mathcal{H}_1)$

For the current Hypothesis testing problem

$$E(T;\mathcal{H}_0) = 0 \ ; \ E(T;\mathcal{H}_1) = A$$

$$var(T;\mathcal{H}_0) = \frac{\sigma^2}{N}$$

$$\therefore d^2 = \frac{A^2}{\sigma^2/N} = \frac{NA^2}{\sigma^2}$$

Since $d^2 = NA^2/\sigma^2$

(i) Detection performance improves as SNR (A^2/σ^2) increases or 'N' the data record length increases.

(ii) Asymptotic Analysis (As $N \to \infty$) proves to be useful

iii) If $w[n] \sim$ IID (Non gaussian noise), then 'T' would not be gaussian; However as $N \to \infty$ we can use CLT to justify 'T' is gaussian. and use first 2 moments to find det. perf.

① ✓ Gaussian
② ✓ Rician
③ ✓ Monte Carlo

Chi Squared ✗
Asymptotic
Quadratic forms of Gaussian } Assignment / tutorial ?

- Gaussian pdf for a scalar RV 'x'

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] ; -\infty < x < \infty$$

denoted by $x \underbrace{\sim}_{\text{distributed according to}} \mathcal{N}(\mu, \sigma^2)$

If $\mu = 0$,

$$E(x^n) = \begin{cases} 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n ; & \text{'n' even} \\ 0 ; & \text{'n' odd} \end{cases}$$

Else

$$E\left[(x+\mu)^n\right] = \sum_{k=0}^{n} \binom{n}{k} E(x^k)\, \mu^{n-k}$$

For $\mu = 0, \sigma^2 = 1$ (Standard Normal pdf / CDF)

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2\right) dt$$

-

- Complementary CDF (Right tail probability)

$Q(x) = 1 - \phi(x)$; $Q(x) = \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{1}{2}x^2\right) \rightarrow$ (a)

(a) is an approximation to $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2\right) dt$

useful when trying to determine if a random variable has a pdf that is approximately Gaussian.

Non Gaussian pdf

$$\phi(x) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) + \frac{1}{2} \frac{1}{\sqrt{2\pi \cdot 2}} \exp\left(-\frac{1}{2 \cdot 2} x^2\right)$$

- Discuss plots of $Q(x)$ here

-

- Multivariate Gaussian pdf of a vector x $(n \times 1)$

$$p(x) = \frac{1}{(2\pi)^{n/2} \det^{1/2}(C)} \exp\left[-\frac{1}{2}(x-\mu)^T C^{-1}(x-\mu)\right]$$

denoted as $x \sim \mathcal{N}(\mu, C)$, assuming C^{-1} exists

and

Mean vector $[\mu]_i = E(x_i)$; $i = 1, 2, \ldots, n$.

cov. Matrix $[C]_{ij} = E\left[(x_i - E(x_i))(x_j - E(x_j))\right]$

; $i = 1, 2, \ldots, n$.
$j = 1, 2, \ldots, n$.

$$C = E\left[(x - E(x))(x - E(x))^T\right].$$

-

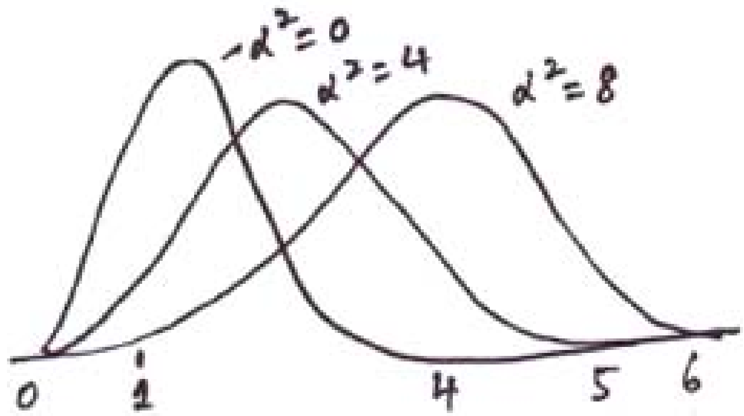
Rician pdf

pdf of $x = \sqrt{x_1^2 + x_2^2}$, where $x_1 \sim \mathcal{N}(\mu_1, \sigma^2)$ and $x_2 \sim \mathcal{N}(\mu_2, \sigma^2)$, and x_1, x_2 are independent

$$p(x) = \begin{cases} x/\sigma^2 \exp\left[-\frac{1}{2\sigma^2}(x^2+\alpha^2)\right] I_0\left(\frac{\alpha x}{\sigma^2}\right); & x > 0 \\ 0 ; & x < 0 \end{cases}$$

and $\alpha^2 = \mu_1^2 + \mu_2^2$ and $I_0(u) = \frac{1}{2\pi}\int_0^{\pi} \exp(u\cos\theta)\, d\theta$.

$$= \int_0^{2\pi} \exp(u\cos\theta) \frac{d\theta}{2\pi}.$$



Right tail probability (Probability of exceeding a certain value)

$$\Pr\{x > \sqrt{\gamma'}\} = \Pr\left\{ \sqrt{\frac{x_1^2 + x_2^2}{\sigma^2}} > \sqrt{\frac{\gamma'}{\sigma^2}} \right\}$$

$$= \Pr\left\{ \frac{x_1^2 + x_2^2}{\sigma^2} > \frac{\gamma'}{\sigma^2} \right\}$$

$$= \underbrace{Q_{\chi'^2_2}}_{\text{Non central } \chi^2 \text{ random variable.}} (\lambda) \left(\frac{\gamma^2}{\sigma^2} \right); \quad \text{where } \lambda = \frac{\mu_1^2 + \mu_2^2}{\sigma^2}$$

Monte Carlo performance Simulation

Problem: $\Pr(RV > \gamma) = \Pr(T > \gamma)$ where T is some statistical description

Methods: (i) Numerical Evaluation.
(ii) Closed form expression.

If (i) and (ii) not possible then resort to Monte Carlo Simulation

eg: Evaluate $\Pr\left\{ \frac{1}{N} \sum_{n=0}^{N-1} x(n) > \gamma \right\}$

$x(n) = \{x[0], \cdots, x[N-1]\}$, $x(n) \sim \mathcal{N}(0, \sigma^2)$ and IID

statistical description $T = 1/N \sum_{n=0}^{N-1} x[n] \sim \mathcal{N}(0, \sigma^2/N)$

$\therefore \quad Pr\{T > \gamma\} = Q\left(\frac{\gamma}{\sqrt{\sigma^2/N}}\right)$

What if we cant evaluate a closed form expression.

MC Simulation:

I. Data Generation

(a) Generate N independent $\mathcal{N}(0,\sigma^2)$ RV's
$[N \times 1]$ column vector

(b) compute $T = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$

(c) Repeat procedure 'M' times, Realize $\{T_1, T_2, \cdots, T_M\}$

II. Probability Evaluation

(i) count number of T_i's that exceed γ
Let this be M_γ

(ii) Estimate $Pr\{T > \gamma\}$ as $\boxed{\hat{P} = \frac{M_\gamma}{M}}$

Larger 'M' ensures convergence in some form

select : $$M \geq \frac{[Q^{-1}(\alpha/2)]^2 (1-P)}{\epsilon^2 P}$$

to get a relative absolute error of $\epsilon = \frac{\hat{P} - P}{P}$
for $100(1-\alpha)\%$ of the time.

Statistical Decision theory - I

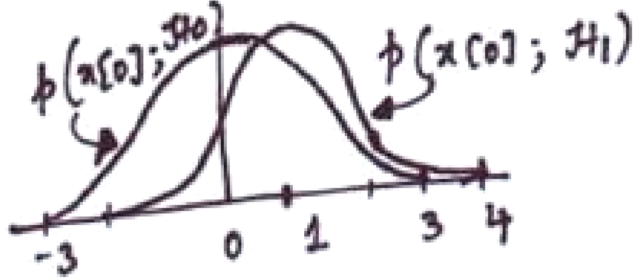
- Simple Hypothesis testing — Neyman Pearson thm. / Bayesian Approach (Bayes Risk)

NP: Sonar Radar ; BR: Communication, Pattern recognition

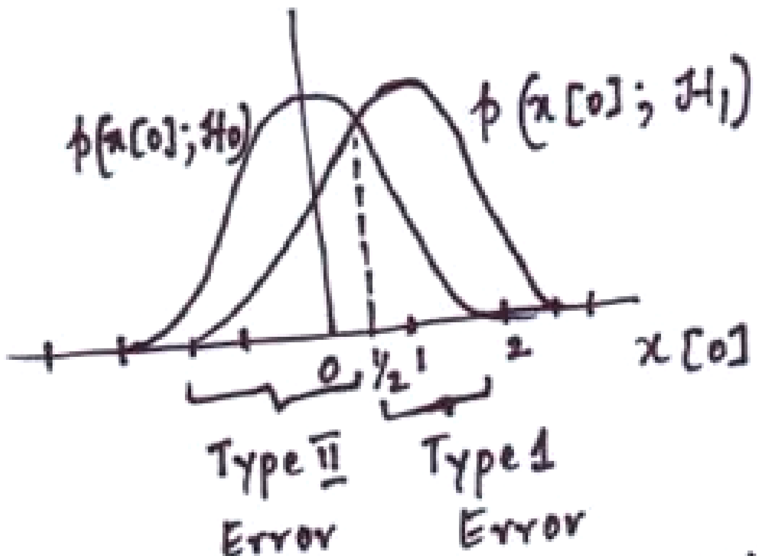
Pbm: Given $x[0]$, Find if $\mathcal{N}(0,1)$ and/or $\mathcal{N}(1,1)$ generated them

Binary Hypothesis

$$\begin{cases} \mathcal{H}_0 : \mu = 0 \ ; & \text{Null Hypothesis} \\ \mathcal{H}_1 : \mu = 1 \ ; & \text{Alternative Hypothesis} \end{cases}$$



-



$p(\underbrace{H_0}_{\text{Decision}} ; \underbrace{H_1}_{\text{True}})$ $\quad p(H_1 ; H_0)$

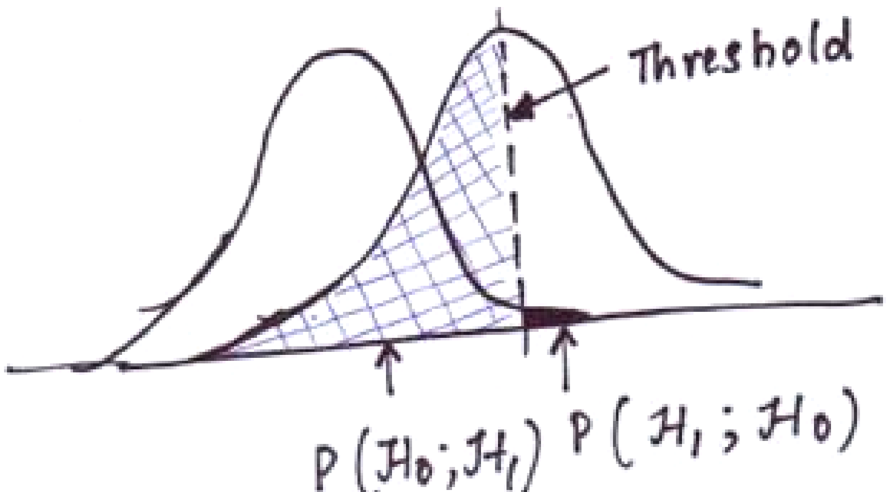
$$\boxed{x[0] > 1/2 \Rightarrow H_1}$$

$\because x[0] > \frac{1}{2} \Rightarrow p(x[0]; H_1)$ is the pdf which prod. $x[0]$

$p(H_i ; H_j)$; Deciding H_i when H_j is true.

Type 1 Error probability: $P(\mathcal{H}_1 ; \mathcal{H}_0)$

Type 2 Error Probability : $P(\mathcal{H}_0 ; \mathcal{H}_1)$



NB: Moving threshold 'DOES NOT' reduce both error probabilities simultaneously (But can constrain one of them to say α)

Eg. Detection problem

$$\mathcal{H}_0 : x[0] = w[0]$$
$$\mathcal{H}_1 : x[0] = s[0] + w[0]$$
$\Big\}$ $s[0] = 1$, $w[0] \sim \mathcal{N}(0,1)$

* Probability of False Alarm $P_{FA} = P(\mathcal{H}_1 ; \mathcal{H}_0)$

<u>Optimal detector</u> : Minimize $P(\mathcal{H}_0 ; \mathcal{H}_1)$

Or : Maximize $[1 - P(\mathcal{H}_0 ; \mathcal{H}_1)]$

* Probability of detection $P_D = P(\mathcal{H}_1 ; \mathcal{H}_1)$

Neyman Pearson Approach (NP) : $\boxed{\text{MAXIMIZE } P_D \text{ SUBJECT TO CONSTRAINT } P_{FA} = P(\mathcal{H}_1 ; \mathcal{H}_0) = \alpha}$

Lets constrain P_{FA} by a threshold γ

$$P_{FA} = P(\mathcal{H}_1; \mathcal{H}_0)$$
$$= P_r\{x[0] > \gamma; \mathcal{H}_0\}$$
$$P_{FA} = \int_{-\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2\right) dt = Q(\gamma)$$

If $P_{FA} = 10^{-3}$, $\gamma = 3$; $\therefore$ Decide $\mathcal{H}_1$ if $x[0] > 3$

Assume $\boxed{\gamma}$

$$\therefore P_D = P_r\{x[0] > \gamma; \mathcal{H}_1\} = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}(t-1)^2\right] dt$$

$$= Q(\gamma - 1) = Q(2) = 0.023.$$

Is P_D the best? May be or NOT (depends on γ)

Detection: Decide either $\mathcal{H}_0$ or $\mathcal{H}_1$ based on observed data $\{x[0], x[1], \dots, x[N-1]\}$

$$\begin{bmatrix} d \\ a \\ t \\ a \end{bmatrix} \xrightarrow{\text{MAP}} \text{Decision region}$$

If $\quad R_1 \in R^N$; $\quad R_1 = \{ x : \text{decide } \mathcal{H}_1 \text{ or reject } \mathcal{H}_0 \}$

$\quad\quad R_0 = \{ x : \text{decide } \mathcal{H}_0 \text{ or reject } \mathcal{H}_1 \}$

R_1 : Critical Region ; R_0 : Complement set of R_1

$\therefore \quad R_0 \cup R_1 = R^N$ <u>Note</u>: Previous Slide $R = \{x[0] > 3\}$

$\therefore$ P_{FA} constraint: $P_{FA} = \int_{R1} p(x; \mathcal{H}_0)\, dx = \alpha$ ← (a)

α : significance Level or size of the test

Many sets R_1 satisfy (a), but we select the one that maximizes $\boxed{P_D = \int_{R1} p(x; \mathcal{H}_1)\, dx}$

P_D is called "Power of the test"

Critical Region that attains maximum power is "best critical region"

"NP THEOROEM": How to choose R_1, given $p(x; \mathcal{H}_0)$, and $p(x; \mathcal{H}_1)$

NP Theorem :

Decide on hypothesis H_1 if

Likelihood Ratio

$$L(x) = \frac{p(x; H_1)}{p(x; H_0)} > \gamma \quad ;$$

Also called Likelihood ratio test

by maximizing P_D for a given $P_{FA} = \alpha$

where threshold γ is found from

$$P_{FA} = \int_{\{x : L(x) > \gamma\}} p(x; H_0)\,dx = \alpha .$$

$L(x)$ indicates likelihood of H_1 versus H_0 for each value of 'x'. [Ex. 3.1 in tutorial]

NP Theoroem : eg: DC Level in WGN

$\mathcal{H}_0 : x[n] = w[n] \; ; \; n = 0, 1, \dots, N-1$

$\mathcal{H}_1 : x[n] = A + \underbrace{w[n]}_{\text{WGN with var} = \sigma^2} \; ; \; n = 0, 1, \dots, N-1$

where $s[n] = A$;
for $A > 0$

Roughly speaking : $\left.\begin{array}{l} \mathcal{H}_0 : x \sim \mathcal{N}(0, \sigma^2 I) \\ \mathcal{H}_1 : x \sim \mathcal{N}(A\mathbf{1}, \sigma^2 I) \end{array}\right]$ Test of 'Mean of a MVG

$\therefore \; \mathcal{H}_0 : \mu = 0$;
$\mathcal{H}_1 : \mu = A\mathbf{1}$; '$\mathbf{1}$' is a veetor of all ones

NP Detector decides on $\mathcal{H}_1$ if

$$\frac{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right]}{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right]} > \gamma$$

It simplifies to (After taking 'LOG' on both sides)

$$\frac{A}{\sigma^2}\sum_{n=0}^{N-1}x[n] > \ln\gamma + \frac{NA^2}{2\sigma^2} ;$$

SINCE $A>0$; $\underbrace{\frac{1}{N}\sum_{n=0}^{N-1}x[n]}_{\text{SAMPLE MEAN}} > \frac{\sigma^2}{NA}\ln\gamma + \frac{A}{2} = \gamma'$

SAMPLE MEAN > THRESHOLD (γ')

IF THE ESTIMATE Mean $\bar{x}$ is "large and (+)ve", then SIGNAL MUST BE PRESENT

HOW LARGE? IS → DEPENDS ON 'NOISE'

∴ ADJUST γ' to control P_{FA}

Make $\boxed{\gamma' \propto \frac{1}{P_{FA}}}$ or $\propto \frac{1}{P_D}$

DEFINE TEST STATISTIC $T(x) = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$ as Gaussian

∴ $E(T(x); \mathcal{H}_0) = E\left(\frac{1}{N}\sum_{n=0}^{N-1} w[n]\right) = 0.$

$E(T(x); \mathcal{H}_1 = E(\frac{1}{N}\sum_{n=0}^{N-1} A + w[n]) = A.$] Mean

$\text{var}(T(x); \mathcal{H}_0) = \frac{\sigma^2}{N}$; $\text{var}(T(x); \mathcal{H}_1) = \frac{\sigma^2}{N}$;

$$\therefore\ T(x) \sim \begin{cases} \mathcal{N}(0, \sigma^2/N) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(A, \sigma^2/N) & \text{under } \mathcal{H}_1 \end{cases}$$

and

$$P_{FA} = P_r\{T(x) > \gamma' ; \mathcal{H}_0\} = Q\left(\frac{\gamma'}{\sqrt{\sigma^2/N}}\right)$$

$$P_D = P_r\{T(x) > \gamma' ; \mathcal{H}_1\} = Q\left(\frac{\gamma' - A}{\sqrt{\sigma^2/N}}\right)$$

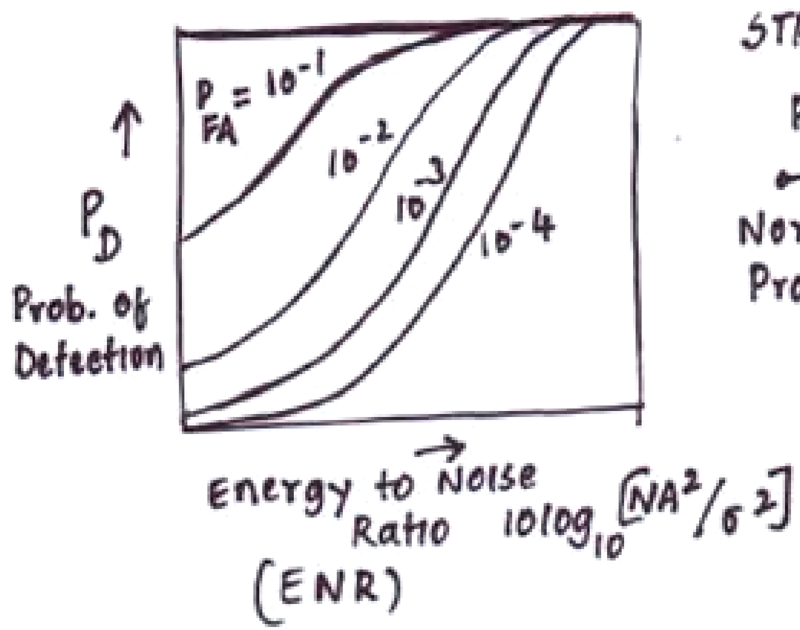
Since Q is monotonically 'INC'

As $(1-Q)$ = cdf which monotonically 'dEC'

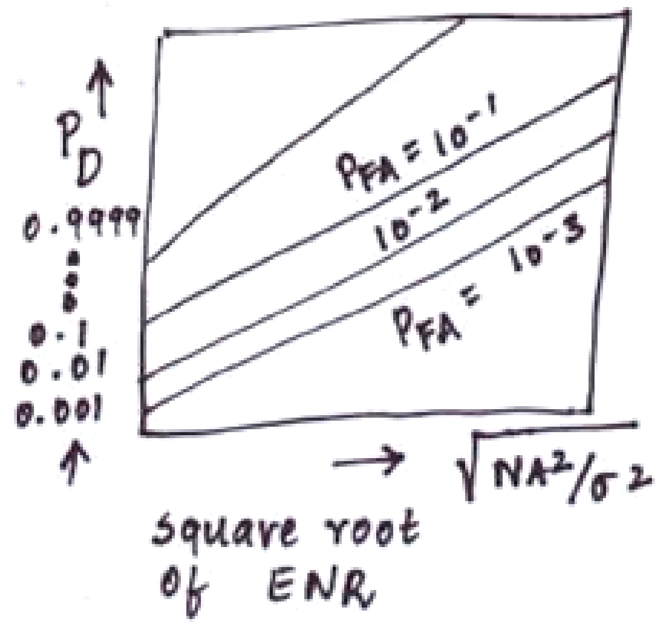
$\boxed{Q^{-1} \text{ exists}}$ ✓ $\quad \therefore\ \gamma' = \sqrt{\frac{\sigma^2}{N}}\, Q^{-1}(P_{FA})$

$$\therefore\ P_D = Q\left(\frac{\sqrt{\sigma^2/N}\, Q^{-1}(P_{FA}) - A}{\sqrt{\sigma^2/N}}\right) = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{NA^2}{\sigma^2}}\right)$$

Detection Performance Curves



TO STRAIGHTEN OUT
Plot on a
$\longrightarrow$
Normal Prob. paper



* One can look at this as a Mean Shifted Gauss-Gauss Problem

$$\boxed{T \sim \begin{cases} \mathcal{N}(\mu_0, \sigma^2) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\mu_1, \sigma^2) & \text{under } \mathcal{H}_1 \end{cases}} \rightarrow \text{where } \boxed{\mu_1 > \mu_0}$$

] Decision based on Mean Shift of 'T'

Deflection Co.efficient (as measure/characteristic of P_D)

$$d^2 = \frac{E(T;\mathcal{H}_1) - E(T;\mathcal{H}_0)}{var(T;\mathcal{H}_0)} = \frac{(\mu_1 - \mu_0)^2}{\sigma^2}$$

When $\mu_0 = 0$, $d^2 = \frac{\mu_1^2}{\sigma^2}$ is 'SNR'.

Start with

$$P_{FA} = P_r\{T > \gamma'; \mathcal{H}_0\} = Q\left(\frac{\gamma' - \mu_0}{\sigma}\right)$$

and $P_D = P_r\{T > \gamma'; \mathcal{H}_1\} = Q\left(\frac{\gamma' - \mu_1}{\sigma}\right) = Q\left(\frac{\mu_0 + \sigma Q^{-1}(P_{FA}) - \mu_1}{\sigma}\right)$

$$P_D = Q\left(Q^{-1}(P_{FA}) - \left(\frac{\mu_1 - \mu_0}{\sigma}\right)\right)$$

$$\boxed{\therefore P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{d^2}\right)} \Longrightarrow \boxed{P_D \propto d}$$

Change in variance to distinguish between $\mathcal{H}_0 / \mathcal{H}_1$

Let $\quad x[n] \sim \mathcal{N}(0, \sigma_0^2)$ under $\mathcal{H}_0$

$\quad\quad x[n] \sim \mathcal{N}(0, \sigma_1^2)$ under $\mathcal{H}_1$

where $\quad \sigma_1^2 > \sigma_0^2$.

NP Test to decide $\mathcal{H}_1$ if

$$\frac{\dfrac{1}{(2\pi\sigma_1^2)^{N/2}} \exp\left(-\dfrac{1}{2\sigma_1^2} \sum_{n=0}^{N-1} x^2[n]\right)}{\dfrac{1}{(2\pi\sigma_0^2)^{N/2}} \exp\left(-\dfrac{1}{2\sigma_0^2} \sum_{n=0}^{N-1} x^2[n]\right)} > \gamma$$

Log on both sides results in

$$-\frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right) \sum_{n=0}^{N-1} x^2[n] > \ln\gamma + \frac{N}{2} \ln \frac{\sigma_1^2}{\sigma_0}$$

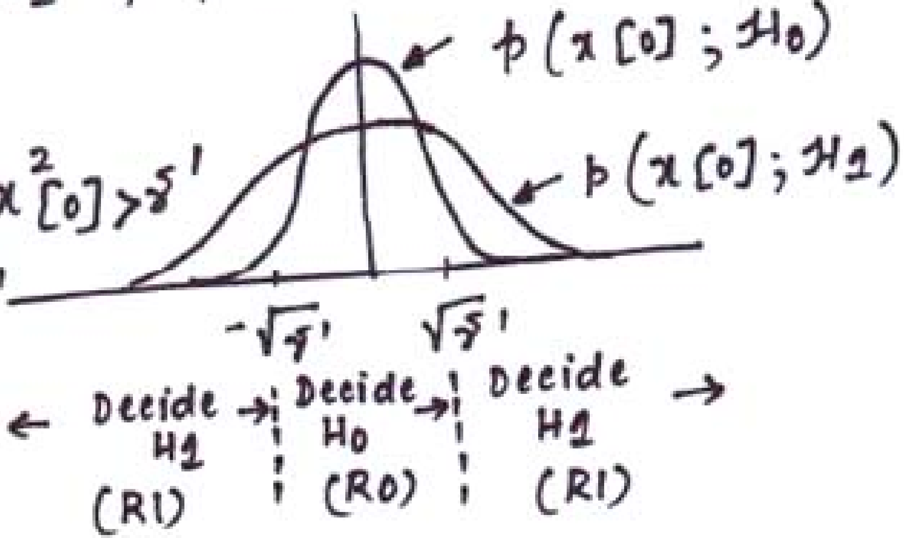
since $\sigma_1^2 > \sigma_0^2$ $\quad \frac{1}{N}\sum_{n=0}^{N-1} x^2[n] > \gamma'$

where $\gamma' = \dfrac{2/N \ln \gamma + \ln \sigma_1^2/\sigma_0^2}{1/\sigma_0^2 - 1/\sigma_1^2}$

$\therefore$ Decide $\mathcal{H}_1$ if power of samples is large enough

Decide $\mathcal{H}_1$; if $x^2[0] > \gamma'$

; if $|x[0]| > \sqrt{\gamma'}$



NP Test and sufficient statistic

$x = [x[0], \cdots, x[N-1]]^T$; pdf $p(x;\theta)$

DC Level in WGN case $\theta = A$

Define $\mathcal{H}_0 : \theta = \theta_0$

$\mathcal{H}_1 : \theta = \theta_1$

use Neyman-Fisher Factorization theoroem and

$$p(x;\theta) = g\left(T(x), \theta\right) h(x)$$

where $\underline{T(x)}$ is sufficient statistic for θ

$\therefore$ NP Test $\frac{p(x;\theta_1)}{p(x;\theta_0)} > \gamma \rightarrow \frac{g(T(x),\theta_1)}{g(T(x),\theta_0)} > \gamma$

NP Test depends on data only thro $T(x)$

$T(x) = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$; DC Level in WGN Case

$T(x) = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]$; change in variance case

* $T(x)$ summarizes all relevant info about θ that is present in the data for a decision.
* If $T(x)$ is unbiased estimator of θ, then detector based on the estimate of θ.

SUFFICIENT STATISTIC DO NOT ALWAYS EXIST

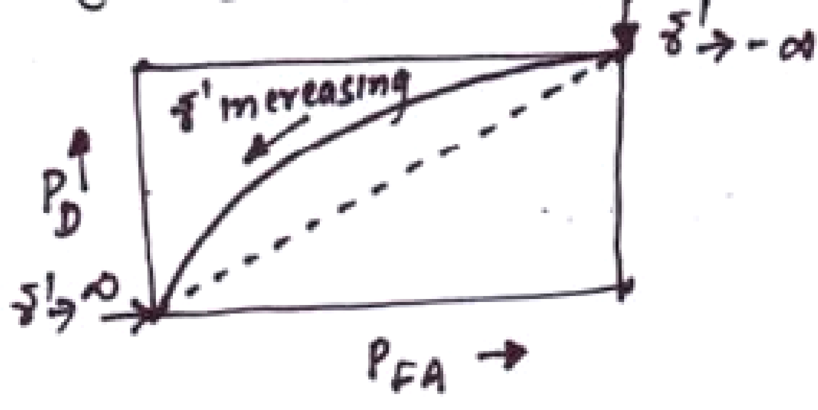
Reciever Operating Characteristics (ROC)

Probability of Detection versus Probability of Acceptance

For the eg: $x[n] = A + w[n]$

$$* \; P_{FA} = Q\left(\frac{\gamma'}{\sqrt{\sigma^2/N}}\right) \; ; \; * \; P_D = Q\left(\frac{\gamma' - A}{\sqrt{\sigma^2/N}}\right)$$

$$* \; P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{d^2}\right) \; ; \; * \; d^2 = \frac{NA^2}{\sigma^2}$$

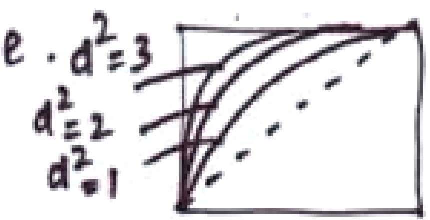


ROC should always be above 45° line (dotted)
Why? Consider detector based on coin flip

$\mathcal{H}_1$: {Head} with $\underline{Pr(\text{Head}) = p}$
$\mathcal{H}_0$: {Tail}

$\therefore P_{FA} = Pr\{\text{head}; \mathcal{H}_0\}$; $P_D = Pr\{\text{head}; \mathcal{H}_1\}$

However $P_{FA} = P_D = p$; Detector generates pts. (p, p)
where $0 \leq p \leq 1$; Hence the 45° line.



<u>AS</u> $d^2 \to \infty$: ideal ROC ($P_D = 1, \forall P_{FA}$)
$d^2 \to 0$: 45° lower bound attained

Minimum Probability of Error

Assign probabilities to Hypothesis

Say $P(\mathcal{H}_0) = P(\mathcal{H}_1) = 1/2$ = Prior probabilities

Eg. Txn. of a 0/1 in Communication

Bayesian Paradigm : Assigns priors

Probability of Error $P_e = Pr\{\text{decide } \mathcal{H}_0, \mathcal{H}_1 \text{ true}\}$
$+ Pr\{\text{decide } \mathcal{H}_1, \mathcal{H}_0 \text{ true}\}$

$$P_e = P(\mathcal{H}_0|\mathcal{H}_1)\, P(\mathcal{H}_1) + P(\mathcal{H}_1|\mathcal{H}_0)\, P(\mathcal{H}_0)$$

$P(\mathcal{H}_i|\mathcal{H}_j)$: Conditional Probability
: "Prob." of deciding $\mathcal{H}_i$, when $\mathcal{H}_j$ is true

Goal : Design a detector that minimizes P_e

General Bayesian Detector decides on $\mathcal{H}_1$ if

$$\underbrace{\frac{p(x|\mathcal{H}_1)}{p(x|\mathcal{H}_0)}}_{\substack{\text{conditional} \\ \text{Likelihood ratio}}} > \underbrace{\frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)}}_{\substack{\text{Ratio} \\ \text{of priors}}} = \gamma$$

If priors equal Then

DECIDE H1 if

$$p(x|\mathcal{H}_1) > p(x|\mathcal{H}_0)$$

$$\boxed{\begin{array}{c}\text{Choose the Hypothesis that maximizes } p(x|\mathcal{H}_i), \text{ for } i = 0,1 \\ \text{MAX. LIKELIHOOD DETECTOR}\end{array}}$$

Eg. DCL in WGN : Minimum P_e Criterion

$\mathcal{H}_0 : x[n] = w[n]$; $\quad n = 0, 1, \ldots, (N-1)$

$\mathcal{H}_1 : x[n] = A + w[n]$; $A > 0$, $w[n]$ is WGN with σ^2 var.

So $x_0[n] = 0$, $x_1[n] = A$ [On-Off keyed commn. System]

$\therefore P(\mathcal{H}_0) = P(\mathcal{H}_1) = 1/2$

For minimum P_e, $\gamma = 1$

OR

$$\frac{\dfrac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\dfrac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - A)^2\right]}{\dfrac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\dfrac{1}{2\sigma^2} \sum_{n=0}^{N-1} x^2[n]\right]} > 1$$

Log on both sides gives

$$-\frac{1}{2\sigma^2}\left(-2A\sum_{n=0}^{N-1} x[n] + NA^2\right) > 0$$

$$\boxed{\text{Decide } \mathcal{H}_1 \text{ if } \bar{x} > A/2}$$

But.

$$\bar{x} \sim \begin{cases} \mathcal{N}(0, \sigma^2/N) \text{ ; conditioned on } \mathcal{H}_0 \\ \mathcal{N}(A, \sigma^2/N) \text{ ; conditioned on } \mathcal{H}_1 \end{cases}$$

Thus

$$P_e = \frac{1}{2}\left[P(\mathcal{H}_0|\mathcal{H}_1) + P(\mathcal{H}_1|\mathcal{H}_0)\right]$$

$$P_e = \frac{1}{2}\left[P_r\{\bar{x} < A/2 | \mathcal{H}_1\} + P_r\{\bar{x} > A/2 | \mathcal{H}_0\}\right]$$

$$P_e = \frac{1}{2}\left[\left(1 - Q\left(\frac{A/2 - A}{\sqrt{\sigma^2/N}}\right)\right) + Q\left(\frac{A/2}{\sqrt{\sigma^2/N}}\right)\right]$$

But $\quad Q(-x) = 1 - Q(x)$

$$P_e = Q\left(\sqrt{\frac{NA^2}{4\sigma^2}}\right);$$ Note NA^2/σ^2 is 'd'

$\therefore \quad P_e \propto 1/d$

'P_e' monotonically decreases with the deflection co.efficient.

Minimum 'P_e' detector (Alternative form)
MAP

Consider $\dfrac{p(x|\mathcal{H}_1)}{p(x|\mathcal{H}_0)} > \dfrac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)} = \gamma$

Decide H1 if

$$p(x|\mathcal{H}_1)\,P(\mathcal{H}_1) > p(x|\mathcal{H}_0)\,P(\mathcal{H}_0)$$

From Bayes Rule

$$P(\mathcal{H}_i|x) = \frac{p(x|\mathcal{H}_i)\,P(\mathcal{H}_i)}{p(x)}$$

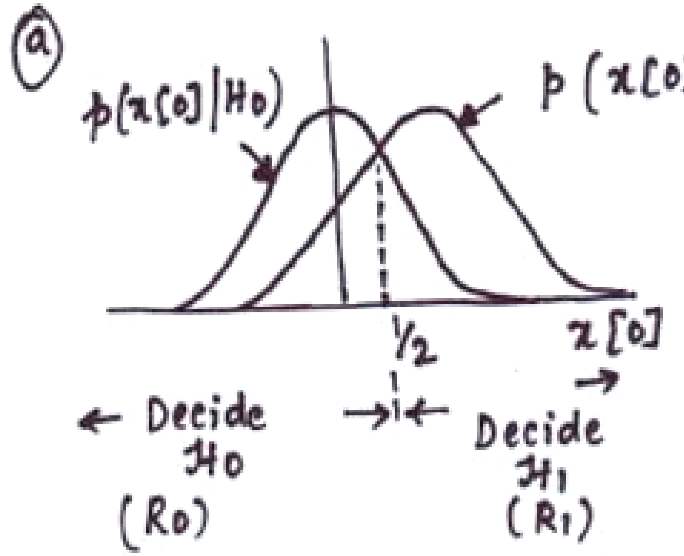
and $p(x) = p(x|\mathcal{H}_0)\,P(\mathcal{H}_0) + p(x|\mathcal{H}_1)\,P(\mathcal{H}_1)$

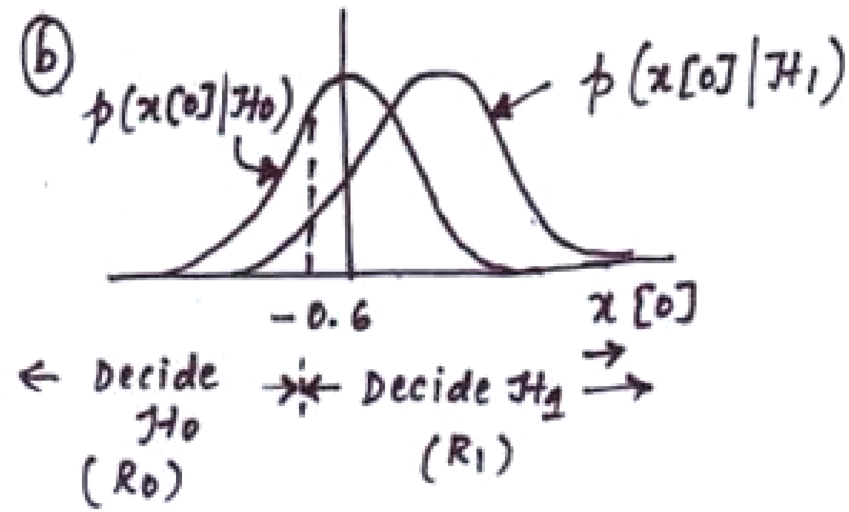
$\therefore$ IF $P(\mathcal{H}_1|x) > P(\mathcal{H}_0|x)$, then Decide "$\mathcal{H}1$".

Previous slide detector $\rightarrow$ MAP detector

$\because$ It minimizes P_e for any prior probability

When Priors are equal ; MAP detector $\rightarrow$ ML detector





Decision regions for (a) $P(\mathcal{H}_0) = P(\mathcal{H}_1) = 1/2$ (b) $P(\mathcal{H}_0) = 1/4$, $P(\mathcal{H}_1) = 3/4$

BAYES RISK

Eg: Inspection of a part $\quad H_0$: part is defective
H_1 : part is satisfactory

Let "C_{ij}" be the cost, when we decide H_i but H_j is true

Generally we should keep $\boxed{C_{10} > C_{01}}$

$\therefore$ Decision Rule minimizes the <u>expected Cost</u>
Bayes Risk (R)

$$\boxed{R = E(C) = \sum_{i=0}^{1} \sum_{j=0}^{1} C_{ij} \, P(H_i|H_j) \, P(H_j)}$$

If no error is made, $\boxed{C_{00} = C_{11} = 0}$

NB: If $C_{00} = C_{11} = 0$, and $C_{10} = C_{01} = 1$

Then $R = P_e$

$\therefore$ Bayes Risk minimization detector

Decides $\mathcal{H}_1$ if

$$\frac{p(x|\mathcal{H}_1)}{p(x|\mathcal{H}_0)} > \frac{C_{10} - C_{00}}{C_{01} - C_{11}} \frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)} = \gamma$$

Assuming $C_{10} > C_{00}$, $C_{01} > C_{11}$

Multiple Hypothesis Testing

'M' signal detection, M-pattern Classification

consider 'M' possible Hypothesis $\{\mathcal{H}_0, \mathcal{H}_1, \cdots, \mathcal{H}_{M-1}\}$

BAYES RISK

$$R = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} c_{ij}\, P(\mathcal{H}_i | \mathcal{H}_j)\, P(\mathcal{H}_j)$$

$$\therefore \text{For } c_{ij} = \begin{cases} 0, & i = j \\ 1, & i \neq j \end{cases} \Rightarrow \boxed{R = P_e} \rightarrow (a)$$

Decision rule that minimizes 'R' above, chooses to minimize the hypothesis

$$C_i(x) = \sum_{j=0}^{M-1} c_{ij}\, P(\mathcal{H}_j | x) \; ; \; i = 0, 1, \cdots, M-1$$

using ⓐ in previous slide

$$C_i(x) = \sum_{\substack{j=0 \\ j \neq i}}^{M-1} P(H_j|x) = \sum_{j=0}^{M-1} \underbrace{P(H_j|x)}_{\text{independent of } 'i'} - P(H_i|x)$$

$\therefore$ $C_i(x)$ is minimized by maximizing $P(H_i|x)$

Minimum P_e decision rule decides 'H_k' if

$$\boxed{P(H_k|x) > P(H_i|x) \quad i \neq k}$$

'M'-ary MAP decision rule

If priors are equal, then

$$p(\mathcal{H}_i|x) = \frac{p(x|\mathcal{H}_i)\,P(\mathcal{H}_i)}{p(x)} = \frac{p(x|\mathcal{H}_i)\,\frac{1}{M}}{p(x)}$$

$\therefore$ Maximizing $P(\mathcal{H}_i|x) \Rightarrow$ maximizing $p(x|\mathcal{H}_i)$

For equal priors, Decide $\mathcal{H}_k$ if

$$\boxed{p(x|\mathcal{H}_k) > p(x|\mathcal{H}_i), \quad i \neq k} \rightarrow \text{'M'-ary ML-decision rule}$$

OR

Maxing $P(\mathcal{H}_i|x) \Rightarrow$ Maxing $p(x|\mathcal{H}_i)\,P(\mathcal{H}_i)$

($\because p(x)$ does not depend on i)

$\therefore$ MAP rule now maximizes $\boxed{\ln p(x|\mathcal{H}_i) + \ln P(\mathcal{H}_i)}$

Composite Hypothesis testing (CHT)

1) **Simple HT**: pdf^s under both hypothesis are completely known

2) **Composite HT**: pdf^s under hypothesis's must accomodate unknown parameters.

Eg. DC Level in WGN $x(n) = A + w[n]$

$$p(x; A, \mathcal{H}_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right]$$

'A' is not known ⇒ pdf is not completely specified

OR pdf under $\mathcal{H}_1$ belongs to a family of pdf^s one for each value of 'A'

CHT (contd.)

Pdf under $\mathcal{H}_1$ $p(x; A, \mathcal{H}_1)$ is parameterized by 'A'.

<u>steps</u>: (1) Design an NP Test as if 'A' were known
(2) Manipulate the test such that it does not depend on 'A'. (Optimal (NP) test)

Eg. DC Level in WGN with Unknown 'A' $(A>0)$

$$\mathcal{H}_0 : x[n] = w[n] \qquad n = 0, 1, \ldots, N-1$$

$$\mathcal{H}_1 : x[n] = A + \underbrace{w[n]}_{A>0,\ \sigma^2 = \text{var.}} \qquad n = 0, 1, \ldots, N-1$$

$$\frac{p(x;A,\mathcal{H}_1)}{p(x;\mathcal{H}_0)} = \frac{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right]}{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right]} > \gamma$$

Taking Log and simplifying we have

$$A\sum_{n=0}^{N-1} x[n] > \sigma^2 \ln\gamma + \frac{NA^2}{2}$$

Since $A > 0$; $\displaystyle\sum_{n=0}^{N-1} x[n] > \frac{\sigma^2}{A}\ln\gamma + \frac{NA}{2}$

Detector

scaling by $1/N$; produces Test $\boxed{T(x) = \frac{1}{N}\sum_{n=0}^{N-1} x[n] > \frac{\sigma^2}{NA}\ln\gamma + \frac{A}{2} = \gamma'}$

Under $\mathcal{H}_0$; $T(x) = \tilde{x} \sim \mathcal{N}(0, \sigma^2/N)$

$\therefore$ $P_{FA} = P_r \{ T(x) > \gamma' ; \mathcal{H}_0 \} = Q\left(\frac{\gamma'}{\sqrt{\sigma^2/N}} \right)$

such that $\gamma' = \sqrt{\frac{\sigma^2}{N}} \, Q^{-1}(P_{FA})$; $\rightarrow$ independent of 'A'

$\therefore$ The Test is an 'NP' detector.

Let see $P_D = ?$; $P_D = P_r \{ T(x) > \gamma' ; \mathcal{H}_1 \}$

But under H_1, $T(x) = \tilde{x} \sim \mathcal{N}(A, \sigma^2/N)$, so that

$$P_D = Q\left(\frac{\gamma' - A}{\sqrt{\sigma^2/N}} \right) = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{NA^2}{\sigma^2}} \right)$$

$\boxed{\therefore P_D \propto A}$

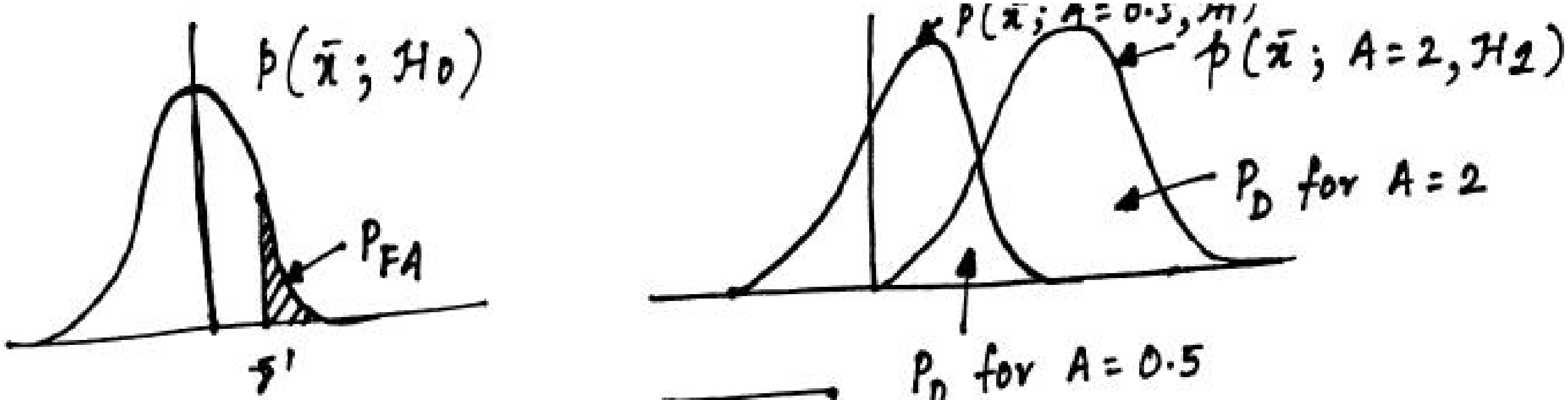


Illustration showing $\boxed{P_D \propto A}$

UMP Test : $T(x)$ that decides $\mathcal{H}_1$ if

$$\frac{1}{N}\sum_{n=0}^{N-1} x[n] > \sqrt{\frac{\sigma^2}{N}}\, Q^{-1}(P_{FA})$$

and yeilds the highest P_D for any value of A, $\forall\, A>0$

UMP Test is done over all possible detectors that have a given P_{FA}; What if $A<0$; UMP Test decides on $\mathcal{H}_1$ if

$$\frac{1}{N}\sum_{n=0}^{N-1} x[n] < -\sqrt{\frac{\sigma^2}{N}}\, Q^{-1}(P_{FA})$$

Recast '$\mathcal{H}$' testing problem, as "parameter testing" problem

$$\underbrace{\left.\begin{array}{l} \mathcal{H}_0 : A = 0 \\ \mathcal{H}_1 : A > 0 \end{array}\right\} \text{One sided Test}}_{\text{May produce UMP Test}} \qquad \underbrace{\left.\begin{array}{l} \mathcal{H}_0 : A = 0 \\ \mathcal{H}_1 : A \neq 0 \end{array}\right\} \text{Two sided Test}}_{\text{Never produce UMP Test}}$$

If UMP Test does not exist, then use suboptimal test (test that has performance close to 'NP' detector)

UMP test analogous to CRLB!!

Clairvoyant Detector: Detector that assumes perfect knowledge of unknown parameter.

Composite Hypothesis Testing

a) Bayesian Approach; (Requires prior PDF) requires multidimensional integration

b) Generalized Likelihood ratio test (GLRT); (does not require prior PDF)

General Problem : Decide between $\mathcal{H}_0$ and $\mathcal{H}_1$

under $\mathcal{H}_0$, vector parameter θ_0 unknown

under $\mathcal{H}_1$, vector parameter θ_1 unknown

Bayesian Approach (Assigns prior pdf's to θ_0 & θ_1)

$$p(x; \mathcal{H}_0) = \int p(x \mid \theta_0 ; \mathcal{H}_0)\, p(\theta_0)\, d\theta_0$$

$$p(x; \mathcal{H}_1 = \int p(x \mid \theta_1 ; \mathcal{H}_1)\, p(\theta_1)\, d\theta_1$$

In general

$p(x|\theta_i;\mathcal{H}_i)$: PDF of x conditioned on θ_i assuming '$\mathcal{H}_i$' is 'true'.

Optimal NP detector for Bayesian Approach

Decide on $\mathcal{H}_1$ if

$$\frac{p(x;\mathcal{H}_1)}{p(x;\mathcal{H}_0)} = \frac{\int p(x|\theta_1;\mathcal{H}_1)\,p(\theta_1)\,d\theta_1}{\int p(x|\theta_0;\mathcal{H}_0)\,p(\theta_0)\,d\theta_0} > \gamma$$

'$\int$' is multidimensional = dimension of unknown parameter (θ_0, θ_1)

Choice of prior difficult : options

* use non informative prior PDF (as flat as possible)

Eg. sinusoid phase estimation : assume $\phi \sim \mathcal{U}[0, 2\pi]$

or Choose Gaussian PDF : $A \sim \mathcal{N}(0, \sigma_A^2)$ & $\sigma_A^2 \rightarrow \infty$

DC Level in WGN with Unknown amplitude - (Bayesian Approach)

'A' is unknown, $-\infty < A < \infty$

Assign a prior pdf $A \sim \mathcal{N}(0, \sigma_A^2)$, 'A' independent of '$w[n]$'

<u>NOTE</u>: As $\sigma_A^2 \to \infty$, PDF becomes non informative prior.

$\therefore$ Conditional PDF <u>under $\mathcal{H}_1$</u>

$$p(x|A;\mathcal{H}_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right]$$

Under $\mathcal{H}_0$ PDF is known, $\therefore$ with $\theta_1 = A$ the NP detector decides $\mathcal{H}_1$ if :

$$\frac{p(x;\mathcal{H}_1)}{p(x;\mathcal{H}_0)} = \frac{\int_{-\infty}^{\infty} p(x|A;\mathcal{H}_1)\, p(A)\, dA}{p(x;\mathcal{H}_0)} > \gamma$$

But $p(x; \mathcal{H}_1) = \int_{-\infty}^{\infty} p(x|A; \mathcal{H}_1)\, p(A)\, dA$

$$= \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right] \frac{1}{\sqrt{2\pi\sigma_A^2}} \exp\left(-\frac{1}{2\sigma_A^2}A^2\right) dA$$

Let $Q(A) = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2 + \frac{A^2}{\sigma_A^2}$

$$= \frac{1}{\sigma^2}\sum_{n=0}^{N-1} x^2[n] - \frac{2N}{\sigma^2}\bar{x}A + \frac{N}{\sigma^2}A^2 + \frac{A^2}{\sigma_A^2}$$

$$= \underbrace{\left(\frac{N}{\sigma^2} + \frac{1}{\sigma_A^2}\right)}_{1/\sigma_{A|x}^2} A^2 - \frac{2N}{\sigma^2}\bar{x}A + \frac{1}{\sigma^2}\sum_{n=0}^{N-1} x^2[n]$$

$$= \frac{A^2}{\sigma^2_{A|x}} - \frac{2N\sigma^2_{A|x}\bar{x}A}{\sigma^2\sigma^2_{A|x}} + \frac{1}{\sigma^2}\sum_{n=0}^{N-1} x^2[n]$$

$$= \frac{1}{\sigma^2_{A|x}}\left(A - \frac{N\bar{x}\sigma^2_{A|x}}{\sigma^2}\right)^2 - \frac{N^2\bar{x}^2}{\sigma^4}\sigma^2_{A|x} + \frac{1}{\sigma^2}\sum_{n=0}^{N-1} x^2[n]$$

so that $$\frac{p(x;\mathcal{H}_1)}{p(x;\mathcal{H}_0)} = \frac{\frac{1}{(2\pi\sigma^2)^{N/2}}\frac{1}{\sqrt{2\pi\sigma^2_A}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}Q(A)\right]dA}{\frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right)}$$

$$\frac{p(x;\mathcal{H}_1)}{p(x;\mathcal{H}_0)} = \frac{1}{\sqrt{2\pi\sigma^2_A}}\sqrt{2\pi\sigma^2_{A|x}}\exp\left(\frac{N^2\bar{x}^2\sigma^2_{A|x}}{2\sigma_4}\right) > \gamma$$

Taking Log on both sides and retaining only dalā dependent terms we decide $\mathcal{H}_1$ if

$$(\bar{x})^2 > \gamma' \quad \text{or} \quad |\bar{x}| > \sqrt{\gamma'}$$

$\Rightarrow$ No knowledge of σ_A^2 required.

But if we assume $A \sim N(\mu_A, \sigma_A^2)$ the knowledge of both μ_A and σ_A^2 required.

Generalized Likelihood Ratio Test (GLRT)

Replaces unknown parameters by their 'ML' estimates (MLE's)

$\boxed{GLRT = UMP}$ among all Invariant Tests under <u>Asymptotic</u> analysis

GLRT (Decides on $\mathcal{H}_1$) if

$$\boxed{L_G(x) = \frac{p(x; \hat{\theta}_1, \mathcal{H}_1)}{p(x; \hat{\theta}_0, \mathcal{H}_0)} > \gamma}$$

; where $\hat{\theta}_1$ is the MLE of θ_1 assuming $\underbrace{\mathcal{H}_1 \text{ is true}}$

↳ MAXIMIZES NUMERATOR of $L_G(x)$

$\underbrace{\hat{\theta}_0}$ is MLE of θ_0 assuming $\mathcal{H}_0$ is true

↳ Maximizes DENOMINATOR of $L_G(x)$

GLRT Also provides info on unknown parameters.

Eg: DC Level in WGN with unknown Amplitude (GLRT)

Assume: $\theta_1 = A$, no unknown parameters under $\mathcal{H}_0$

Hypothesis Test

$$\left.\begin{aligned} \mathcal{H}_0 &: A = 0 \\ \mathcal{H}_1 &: A \neq 0 \end{aligned}\right\}$$

GLRT decides $\mathcal{H}_1$ if

$$L_G(x) = \frac{p(x; \hat{A}, \mathcal{H}_1)}{p(x; \mathcal{H}_0)} > \gamma$$

Find MLE of 'A' by maximizing

$$p(x; A, \mathcal{H}_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - A)^2\right]$$

which yeilds $\boxed{\hat{A} = \bar{x}}$

$$\therefore \quad L_G(x) = \frac{\dfrac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\dfrac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\bar{x})^2\right]}{\dfrac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\dfrac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right)}$$

$$\ln L_G(x) = -\frac{1}{2\sigma^2}\left[\sum_{n=0}^{N-1}x^2[n] - 2\bar{x}\sum_{n=0}^{N-1}x[n] + N\bar{x}^2 - \sum_{n=0}^{N-1}x^2[n]\right]$$

$$\ln L_G(x) = \frac{N\bar{x}^2}{2\sigma^2} \; ;$$

$\therefore$ Decide $\mathcal{H}_1$ if $\boxed{|\bar{x}| > \gamma'}$

Alternative form of GLRT

Since $\hat{\theta}_i$ is MLE under $\mathcal{H}_i$

$$p(x; \hat{\theta}_i, \mathcal{H}_i) = \max_{\theta_i} p(x; \theta_i, \mathcal{H}_i)$$

$$\therefore L_G(x) = \frac{\max_{\theta_1} p(x; \theta_1, \mathcal{H}_1)}{\max_{\theta_0} p(x; \theta_0, \mathcal{H}_0)}$$

For case where PDF under $\mathcal{H}_0$ is completely known

$$L_G(x) = \frac{\max_{\theta_1} p(x; \theta_1, \mathcal{H}_1)}{p(x; \mathcal{H}_0)}$$

$$L_G(x) = \max_{\theta_1} \frac{p(x;\theta_1,\mathcal{H}_1)}{p(x;\mathcal{H}_0)}$$

<u>OR</u> Maximize Likelihood ratio over θ_1, so that

$$\boxed{L_G(x) = \max_{\theta_1} L(x;\theta_1)}$$